Vector analysis 1

Frames 1 to 95

Learning outcomes

When you have completed this Programme you will be able to:

- Obtain the scalar and vector product of two vectors
- Reproduce the relationships between the scalar and vector products of the Cartesian coordinate unit vectors
- Obtain the scalar and vector triple products and appreciate their geometric significance
- Differentiate a vector field and derive a unit vector tangential to the vector field at a point
- Integrate a vector field
- Obtain the gradient of a scalar field, the directional derivative and a unit normal to a surface
- Obtain the divergence of a vector field and recognise a solenoidal vector field
- Obtain the curl of a vector field
- Obtain combinations of div, grad and curl acting on scalar and vector fields as appropriate

Prerequisite: Engineering Mathematics (Fifth Edition)
Programme 6 Vectors

Introduction

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The initial work on vectors was covered in detail in Programme 6 of *Engineering Mathematics (Fifth Edition)* and, if you are in any doubt, spend some time reviewing that section of the work before proceeding further.

The current Programmes on vector analysis build on these early foundations, so, for quick reference, the essential results of the previous work are summarised in the following list.

Summary of prerequisites

1 A *scalar* quantity has magnitude only; a *vector* quantity has both magnitude and direction.

2 The axes of reference, OX, OY, OZ, form a right-handed set. The symbols $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ denote *unit vectors* in the directions OX, OY, OZ, respectively.

If $\overline{\mathrm{OP}} = \mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ then $\mathrm{OP} = |\mathbf{r}| = \sqrt{a_x^2 + a_y^2 + a_z^2}$ where $|\mathbf{r}|$ is the modulus of $\mathbf{r}$.

3 The *direction cosines* $[l, m, n]$ are the cosines of the angles between the vector $\mathbf{r}$ and the axes OX, OY, OZ, respectively. For any vector $\mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

$$l = \frac{a_x}{|\mathbf{r}|}; \quad m = \frac{a_y}{|\mathbf{r}|}; \quad n = \frac{a_z}{|\mathbf{r}|}$$

and $\quad l^2 + m^2 + n^2 = 1.$

4 *Scalar product* ('dot product')

$\mathbf{A} \cdot \mathbf{B} = AB\cos\theta$ where θ is the angle between $\mathbf{A}$ and $\mathbf{B}$ and where A and B are the moduli of $\mathbf{A}$ and $\mathbf{B}$.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ then

$$\mathbf{A} \cdot \mathbf{B} = a_x b_x + a_y b_y + a_z b_z \quad \text{and} \quad \mathbf{A} \cdot \mathbf{B} = \mathbf{B} \cdot \mathbf{A}$$

5 *Vector product* ('cross product')

$\mathbf{A} \times \mathbf{B} = AB\sin\theta$ in a direction perpendicular to $\mathbf{A}$ and $\mathbf{B}$ so that $\mathbf{A}$, $\mathbf{B}$, $(\mathbf{A} \times \mathbf{B})$ form a right-handed set.

Therefore $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$

$$\text{Also} \quad \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \quad \text{where } \mathbf{A} \times \mathbf{B} = -\mathbf{B} \times \mathbf{A}$$

6 *Angle between two vectors*

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2$$

where l_1, m_1, n_1 and l_2, m_2, n_2 are the direction cosines of vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ respectively.

For perpendicular vectors $\quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$

For parallel vectors $\quad l_1 l_2 + m_1 m_2 + n_1 n_2 = 1.$

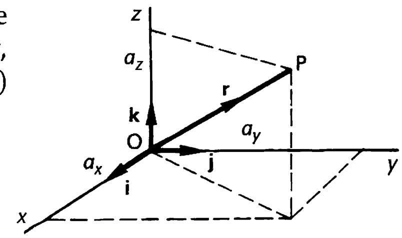
One or two examples will no doubt help to recall the main points. ▶

Example 1 (Direction cosines)

If $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ are unit vectors in the directions OX, OY, OZ, respectively, then any position vector $\overline{\text{OP}}$ (= $\mathbf{r}$) can be represented in the form

$$\overline{\text{OP}} = \mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}.$$

Then $|\mathbf{r}| = \ldots\ldots\ldots\ldots$



2

$$\boxed{|\mathbf{r}| = \sqrt{a_x^2 + a_y^2 + a_z^2}}$$

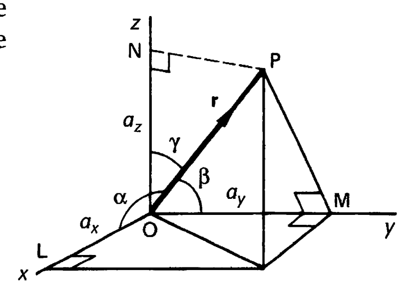
The direction of OP is denoted by stating the direction cosines of the angles made by OP and the three coordinate axes.

$$l = \cos\alpha = \frac{\text{OL}}{\text{OP}} = \frac{a_x}{|\mathbf{r}|}$$

$$m = \cos\beta = \frac{\text{OM}}{\text{OP}} = \frac{a_y}{|\mathbf{r}|}$$

$$n = \cos\gamma = \frac{\text{ON}}{\text{OP}} = \frac{a_z}{|\mathbf{r}|}$$

$$\therefore\; l,\; m,\; n = \cos\alpha,\quad \cos\beta,\quad \cos\gamma$$



So, if P is the point (3, 2, 6), then

$$|\mathbf{r}| = \ldots\ldots\ldots\ldots;$$
$$l = \ldots\ldots\ldots\ldots;\quad m = \ldots\ldots\ldots\ldots;\quad n = \ldots\ldots\ldots\ldots$$

3

$$\boxed{\begin{gathered} |\mathbf{r}| = 7; \\ l = 0{\cdot}429;\quad m = 0{\cdot}286;\quad n = 0{\cdot}857 \end{gathered}}$$

Because

$$(|\,\mathbf{r}\,|)^2 = 9 + 4 + 36 = 49 \qquad \therefore\quad |\,\mathbf{r}\,| = 7$$

$$l = \cos\alpha = \tfrac{3}{7} = 0{\cdot}4286$$

$$m = \cos\beta = \tfrac{2}{7} = 0{\cdot}2857$$

$$n = \cos\gamma = \tfrac{6}{7} = 0{\cdot}8571.$$

▶

Example 2 (Angle between two vectors)

If the direction cosines of $\mathbf{A}$ are l_1, m_1, n_1 and those of $\mathbf{B}$ are l_2, m_2, n_2, then the angle between the vectors is given by

$$\cos\theta = l_1 l_2 + m_1 m_2 + n_1 n_2. \qquad (1)$$

If $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$, we can find the direction cosines of each and hence θ which is ....

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$$\boxed{\theta = 66^\circ\ 36'}$$

Because

For $\mathbf{A}$: $|\,\mathbf{r}_1\,| = \sqrt{4+9+16} = \sqrt{29}$

$$\therefore\ l_1 = \frac{2}{\sqrt{29}};\quad m_1 = \frac{3}{\sqrt{29}};\quad n_1 = \frac{4}{\sqrt{29}}$$

For $\mathbf{B}$: $|\,\mathbf{r}_2\,| = \sqrt{1+4+9} = \sqrt{14}$

$$\therefore\ l_2 = \frac{1}{\sqrt{14}};\quad m_2 = \frac{-2}{\sqrt{14}};\quad n_2 = \frac{3}{\sqrt{14}}$$

Then $\cos\theta = \dfrac{1}{\sqrt{14\times 29}}\{2 - 6 + 12\} = 0{\cdot}3970$

$\therefore\ \theta = 66^\circ\ 36'$

Let us now look at the question of scalar and vector products.

On to the next frame

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Example 3 (Scalar product)

If $\mathbf{A}$ and $\mathbf{B}$ are two vectors, the scalar product of $\mathbf{A}$ and $\mathbf{B}$ is defined as

$$\mathbf{A}\cdot\mathbf{B} = AB\cos\theta \qquad (2)$$

where θ is the angle between the two vectors. If $\mathbf{A}\cdot\mathbf{B} = 0$ then $\mathbf{A}\perp\mathbf{B}$.

If we consider the *scalar products of the unit vectors* $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, which are mutually perpendicular, then

$$\mathbf{i}\cdot\mathbf{j} = (1)(1)\cos 90^\circ = 0 \qquad \therefore\ \mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0$$

and $$\mathbf{i}\cdot\mathbf{i} = (1)(1)\cos 0^\circ = 1 \qquad \therefore\ \mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1.$$

In general, if $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ then $\mathbf{A}\cdot\mathbf{B} = a_x b_x + a_y b_y + a_z b_z$ which is, of course, a scalar quantity.

So, if $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k}$, then

$$\mathbf{A}\cdot\mathbf{B} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\mathbf{A}\cdot\mathbf{B} = 2 - 6 + 20 = 16}$$

Also, since $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$, we can determine the angle θ between the vectors. In this case $\theta = \ldots\ldots\ldots\ldots$

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$$\boxed{\theta = 57^\circ\ 9'}$$

$\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k} \qquad \therefore\ A = |\mathbf{A}| = \sqrt{4+9+16} = \sqrt{29}$

$\mathbf{B} = \mathbf{i} + 2\mathbf{j} + 5\mathbf{k} \qquad \therefore\ B = |\mathbf{B}| = \sqrt{1+4+25} = \sqrt{30}$

We have already found that $\mathbf{A}\cdot\mathbf{B} = 16$ and $\mathbf{A}\cdot\mathbf{B} = AB\cos\theta$

$\therefore\ 16 = \sqrt{29}\,\sqrt{30}\cos\theta \quad \therefore\ \cos\theta = 0{\cdot}5425 \quad \therefore\ \theta = 57^\circ\ 9'$

So, the *scalar product* of $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$

is $\quad \mathbf{A}\cdot\mathbf{B} = a_xb_x + a_yb_y + a_zb_z$

and $\quad \mathbf{A}\cdot\mathbf{B} = AB\cos\theta$ where θ is the angle between the vectors.

It can also be shown that

(a) $\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A}$

and (b) $\mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$

Make a note of these results

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Example 4 (Vector product)

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ the vector product $\mathbf{A}\times\mathbf{B}$ has magnitude $|\mathbf{A}\times\mathbf{B}| = AB\sin\theta$ in the direction perpendicular to $\mathbf{A}$ and $\mathbf{B}$ such that $\mathbf{A}$, $\mathbf{B}$ and $(\mathbf{A}\times\mathbf{B})$ form a right-handed set.

We can write this as

$$\mathbf{A}\times\mathbf{B} = (AB\sin\theta)\mathbf{n} \tag{3}$$

where $\mathbf{n}$ is defined as a unit vector in the positive normal direction to the plane of $\mathbf{A}$ and $\mathbf{B}$, i.e. forming a right-handed set.

Also $$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{4}$$

If we consider the *vector products of the unit vectors*, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$, then

$\mathbf{i}\times\mathbf{j} = (1)(1)\sin 90^\circ\mathbf{k} = \mathbf{k}$

$\mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}$

Note that

$\mathbf{j}\times\mathbf{i} = -(\mathbf{i}\times\mathbf{j}) = -\mathbf{k}$

$\mathbf{k}\times\mathbf{j} = -\mathbf{i}, \quad \mathbf{i}\times\mathbf{k} = -\mathbf{j}$

Also

$\mathbf{i}\times\mathbf{i} = (1)(1)\sin 0^\circ\mathbf{n} = \mathbf{0}$

$\mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$

It can also be shown that

(a) $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$

and (b) $\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$ (5)

Make a note of these results (3), (4) and (5).

Then, if $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}$

$\mathbf{A}\times\mathbf{B} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{A} \times \mathbf{B} = 16\mathbf{i} + 14\mathbf{j} - 5\mathbf{k}}$$

We simply evaluate the determinant

$$\mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -2 & 4 \\ 2 & -3 & -2 \end{vmatrix}$$

$$= \mathbf{i}(4+12) - \mathbf{j}(-6-8) + \mathbf{k}(-9+4) = 16\mathbf{i} + 14\mathbf{j} - 5\mathbf{k}$$

Move on to the next frame

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We have seen therefore that

the scalar product of two vectors is a scalar
but that the vector product of two vectors is a vector.

We know also that $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$

Therefore, the angle between the vectors **A** and **B** given in Example 4 is

$\theta = \ldots\ldots\ldots\ldots$

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$$\boxed{\theta = 79^\circ\ 40'}$$

Because

$$\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}; \quad \mathbf{B} = 2\mathbf{i} - 3\mathbf{j} - 2\mathbf{k}; \quad \text{and} \quad \mathbf{A} \times \mathbf{B} = 16\mathbf{i} + 14\mathbf{j} - 5\mathbf{k}$$

$$\therefore \quad |\mathbf{A} \times \mathbf{B}| = \sqrt{16^2 + 14^2 + 5^2} = \sqrt{477} = 21{\cdot}84$$

$$A = |\mathbf{A}| = \sqrt{3^2 + 2^2 + 4^2} = \sqrt{29} = 5{\cdot}385$$

$$B = |\mathbf{B}| = \sqrt{2^2 + 3^2 + 2^2} = \sqrt{17} = 4{\cdot}123$$

$$\therefore \quad 21{\cdot}84 = (5{\cdot}385)(4{\cdot}123)\sin\theta$$

$$\therefore \quad \sin\theta = 0{\cdot}9838 \quad \therefore \quad \theta = 79^\circ\ 40'$$

So, to recapitulate:

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ and $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$ and θ is the angle between them

(a) Scalar product $= \mathbf{A} \cdot \mathbf{B} = a_xb_x + a_yb_y + a_zb_z$

$$= AB\cos\theta$$

(b) Vector product $= \mathbf{A} \times \mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$

and $|\mathbf{A} \times \mathbf{B}| = AB\sin\theta$.

Make a note of these fundamental results: we shall certainly need them. Then, in the next frame, we can set off on some new work

Triple products

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We now deal with the various products that we form with three vectors.

Scalar triple product of three vectors

If $\mathbf{A}, \mathbf{B}, \mathbf{C}$ are three vectors, the scalar formed by the product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$ is called the scalar triple product.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$; $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$; $\mathbf{C} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$;

then $$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\therefore\ \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}) \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

Multiplying the top row by the external bracket and remembering that

$$\mathbf{i} \cdot \mathbf{j} = \mathbf{j} \cdot \mathbf{k} = \mathbf{k} \cdot \mathbf{i} = 0 \quad \text{and} \quad \mathbf{i} \cdot \mathbf{i} = \mathbf{j} \cdot \mathbf{j} = \mathbf{k} \cdot \mathbf{k} = 1$$

we have $$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix} \qquad (6)$$

Example

If $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$; $\mathbf{B} = \mathbf{i} - 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{C} = 2\mathbf{i} + \mathbf{j} + 2\mathbf{k}$;

then $$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & -2 & -3 \\ 2 & 1 & 2 \end{vmatrix}$$
$$= \ldots\ldots\ldots\ldots$$

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$$\boxed{\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = 42}$$

Because

$$\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 2 & -3 & 4 \\ 1 & -2 & -3 \\ 2 & 1 & 2 \end{vmatrix}$$
$$= 2(-4 + 3) + 3(2 + 6) + 4(1 + 4) = 42$$

As simple as that.

14 Properties of scalar triple products

(a) $$\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\begin{vmatrix} b_x & b_y & b_z \\ c_x & c_y & c_z \\ a_x & a_y & a_z \end{vmatrix}=-\begin{vmatrix} a_x & a_y & a_z \\ c_x & c_y & c_z \\ b_x & b_y & b_z \end{vmatrix}$$

since interchanging two rows in a determinant reverses the sign. If we now interchange rows 2 and 3 and again change the sign, we have

$$\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}=\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$$

$$\therefore\ \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{A})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) \tag{7}$$

i.e. the scalar triple product is unchanged by a cyclic change of the vectors involved.

(b) $$\mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})=\begin{vmatrix} b_x & b_y & b_z \\ a_x & a_y & a_z \\ c_x & c_y & c_z \end{vmatrix}=-\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\therefore\ \mathbf{B}\cdot(\mathbf{A}\times\mathbf{C})=-\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) \tag{8}$$

i.e. a change of vectors not in cyclic order, changes the sign of the scalar triple product.

(c) $$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{A})=\begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ a_x & a_y & a_z \end{vmatrix}=0 \text{ since two rows are identical.}$$

$$\therefore\ \mathbf{A}\cdot(\mathbf{B}\times\mathbf{A})=\mathbf{B}\cdot(\mathbf{C}\times\mathbf{B})=\mathbf{C}\cdot(\mathbf{A}\times\mathbf{C})=0 \tag{9}$$

Example

If $\mathbf{A}=\mathbf{i}+2\mathbf{j}+3\mathbf{k}$; $\mathbf{B}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$; $\mathbf{C}=3\mathbf{i}+\mathbf{j}-2\mathbf{k}$

$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=$ .... $\quad\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})=$ ....

$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = 52; \quad \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B}) = -52}$$

Because

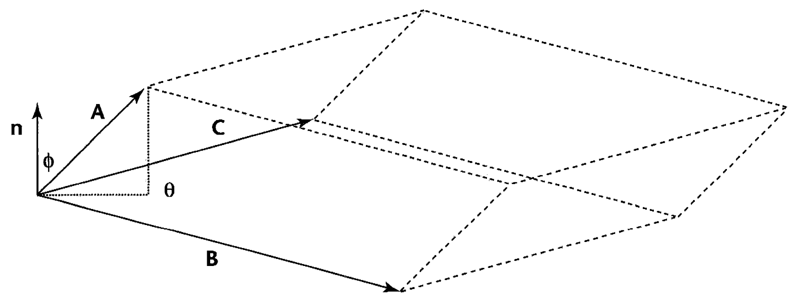
$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} 1 & 2 & 3 \\ 2 & -3 & 1 \\ 3 & 1 & -2 \end{vmatrix} = 1(6-1) - 2(-4-3) + 3(2+9) = 52$$

$\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A})$ is not a cyclic change from the above. Therefore

$$\mathbf{C}\cdot(\mathbf{B}\times\mathbf{A}) = -\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = -52$$

Coplanar vectors

The magnitude of the scalar triple product $|\,\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\,|$ is equal to the volume of the parallelepiped with three adjacent sides defined by **A**, **B** and **C**.



The scalar triple product $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{A}\cdot(BC\sin\theta\,\mathbf{n}) = ABC\sin\theta\cos\phi$ where $\mathbf{n}$ is a unit vector perpendicular to the plane containing **B** and **C**, θ is the angle between **B** and **C** and ϕ is the angle between **A** and $\mathbf{n}$. Therefore

$$|\,\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\,| = ABC\,|\,\sin\theta\cos\phi\,|$$

Notice that in the figure both θ and ϕ are drawn as acute but in the general case this may not be so. Now, $BC\,|\sin\theta\,|$ is the area of the parallelogram defined by **B** and **C**. The altitude of the parallelepiped is $A\,|\cos\phi\,|$ and so $ABC\,|\sin\theta\cos\phi\,|$ is the *volume* of the parallelepiped with three adjacent sides defined by **A**, **B** and **C**.

Consequently if $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{0}$ then the volume of the parallelepiped is zero and the three vectors **A**, **B** and **C** are coplanar.

Example 1

Show that $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 2\mathbf{k}$; and $\mathbf{C} = 3\mathbf{i} + \mathbf{j} - \mathbf{k}$ are coplanar.

We just evaluate $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \ldots\ldots\ldots\ldots$ and apply the test.

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$$\boxed{\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0}$$

Because

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}1 & 2 & -3\\ 2 & -1 & 2\\ 3 & 1 & -1\end{vmatrix}=1(1-2)-2(-2-6)-3(2+3)\ =0.$$

Therefore **A, B, C** are coplanar.

Example 2

If $\mathbf{A}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$; $\mathbf{B}=3\mathbf{i}+2\mathbf{j}+\mathbf{k}$; $\mathbf{C}=\mathbf{i}+p\mathbf{j}+4\mathbf{k}$ are coplanar, find the value of p.

The method is clear enough. We merely set up and evaluate the determinant and solve the equation $\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0$.

$p=$ ....

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$$\boxed{p=-3}$$

Because

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=0 \qquad \therefore \begin{vmatrix}2 & -1 & 3\\ 3 & 2 & 1\\ 1 & p & 4\end{vmatrix}=0$$

$\therefore\ 2(8-p)+1(12-1)+3(3p-2)=0 \quad \therefore\ 7p=-21 \quad \therefore\ p=-3$

One more.

Example 3

Determine whether the three vectors $\mathbf{A}=3\mathbf{i}+2\mathbf{j}-\mathbf{k}$; $\mathbf{B}=2\mathbf{i}-\mathbf{j}+3\mathbf{k}$; $\mathbf{C}=\mathbf{i}-2\mathbf{j}+2\mathbf{k}$ are coplanar.

Work through it on your own. The result shows that

....

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$$\boxed{\mathbf{A},\ \mathbf{B},\ \mathbf{C} \text{ are not coplanar}}$$

Because

$$\text{in this case } \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}3 & 2 & -1\\ 2 & -1 & 3\\ 1 & -2 & 2\end{vmatrix}=13$$

$\therefore\ \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})\neq 0 \qquad \therefore$ **A, B, C** are not coplanar.

Now on to something different

Vector triple products of three vectors

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If **A**, **B** and **C** are three vectors, then

$$\left.\begin{array}{rr} & \mathbf{A}\times(\mathbf{B}\times\mathbf{C}) \\ \text{and} & (\mathbf{A}\times\mathbf{B})\times\mathbf{C}\end{array}\right\}\ \text{are called the vector triple products.} \qquad (10)$$

Consider $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ where $\mathbf{A}=a_x\mathbf{i}+a_y\mathbf{j}+a_z\mathbf{k}$; $\mathbf{B}=b_x\mathbf{i}+b_y\mathbf{j}+b_z\mathbf{k}$ and $\mathbf{C}=c_x\mathbf{i}+c_y\mathbf{j}+c_z\mathbf{k}$.

Then $(\mathbf{B}\times\mathbf{C})$ is a vector perpendicular to the plane of **B** and **C** and $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ is a vector perpendicular to the plane containing **A** and $(\mathbf{B}\times\mathbf{C})$, i.e. coplanar with **B** and **C**.

Note that, similarly, $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$ is coplanar with **A** and **B** and so in general $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})\neq(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$.

Now

$$(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ b_x & b_y & b_z\\ c_x & c_y & c_z\end{vmatrix}=\mathbf{i}\begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix}-\mathbf{j}\begin{vmatrix}b_x & b_z\\ c_x & c_z\end{vmatrix}+\mathbf{k}\begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}$$

Then $$\mathbf{A}\times(\mathbf{B}\times\mathbf{C})=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ \begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix} & -\begin{vmatrix}b_x & b_z\\ c_x & c_z\end{vmatrix} & \begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}\end{vmatrix}$$

$$=\begin{vmatrix}\mathbf{i} & \mathbf{j} & \mathbf{k}\\ a_x & a_y & a_z\\ \begin{vmatrix}b_y & b_z\\ c_y & c_z\end{vmatrix} & \begin{vmatrix}b_z & b_x\\ c_z & c_x\end{vmatrix} & \begin{vmatrix}b_x & b_y\\ c_x & c_y\end{vmatrix}\end{vmatrix}$$

In symbolic form, further expansion of the determinant becomes somewhat tedious. However a numerical example will clarify the method.

So make a note of the definition (10) above and then go on to the next frame

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Example 1

If $\mathbf{A}=2\mathbf{i}-3\mathbf{j}+\mathbf{k}$; $\mathbf{B}=\mathbf{i}+2\mathbf{j}-\mathbf{k}$; $\mathbf{C}=3\mathbf{i}+\mathbf{j}+3\mathbf{k}$; determine the vector triple product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$.

We start off with $\mathbf{B}\times\mathbf{C}=$ ....

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$$\boxed{\mathbf{B} \times \mathbf{C} = 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k}}$$

Because

$$\mathbf{B} \times \mathbf{C} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 1 & 2 & -1 \\ 3 & 1 & 3 \end{vmatrix} \begin{aligned} &= \mathbf{i}(6+1) - \mathbf{j}(3+3) + \mathbf{k}(1-6) \\ &= 7\mathbf{i} - 6\mathbf{j} - 5\mathbf{k} \end{aligned}$$

Then $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \ldots\ldots\ldots\ldots$

22

$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = 21\mathbf{i} + 17\mathbf{j} + 9\mathbf{k}}$$

Because

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & -3 & 1 \\ 7 & -6 & 5 \end{vmatrix} \\ &= \mathbf{i}(15+6) - \mathbf{j}(-10-7) + \mathbf{k}(-12+21) \\ &= 21\mathbf{i} + 17\mathbf{j} + 9\mathbf{k} \end{aligned}$$

That is fundamental enough. There is, however, an even easier way of determining a vector triple product. It can be proved that

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ \text{and} \quad (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \end{aligned} \tag{11}$$

The proof of this is given in the Appendix. For the moment, make a careful note of the expressions: then we will apply the method to the example we have just completed.

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$\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$; $\mathbf{B} = \mathbf{i} + 2\mathbf{j} - \mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ and we have

$$\begin{aligned} \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ &= (6-3+3)(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) - (2-6-1)(3\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= 6\,(\mathbf{i} + 2\mathbf{j} - \mathbf{k}) + 5(3\mathbf{i} + \mathbf{j} + 3\mathbf{k}) \\ &= 21\mathbf{i} + 17\mathbf{j} + 9\mathbf{k} \end{aligned}$$

which is, of course, the result we achieved before.

Here is another.

Example 2

If $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}$; $\mathbf{B} = 4\mathbf{i} - \mathbf{j} + 3\mathbf{k}$; $\mathbf{C} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$ determine $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ using the relationship $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A}$.

$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \ldots\ldots\ldots\ldots$$

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$$\boxed{-50\mathbf{i} - 26\mathbf{j} + 22\mathbf{k}}$$

Because

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \\ &= (6 - 6 - 2)(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) - (8 + 3 + 3)(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \\ &= -2(4\mathbf{i} - \mathbf{j} + 3\mathbf{k}) - 14(3\mathbf{i} + 2\mathbf{j} - 2\mathbf{k}) \\ &= -50\mathbf{i} - 26\mathbf{j} + 22\mathbf{k}\end{aligned}$$

Now one more.

Example 3

If $\mathbf{A} = \mathbf{i} + 3\mathbf{j} + 2\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} + 5\mathbf{j} - \mathbf{k}$; $\mathbf{C} = \mathbf{i} + 2\mathbf{j} + 3\mathbf{k}$

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \dots\dots\dots\dots$$
$$(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = \dots\dots\dots\dots$$

Finish them both.

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$$\boxed{\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= 11\mathbf{i} + 35\mathbf{j} - 58\mathbf{k} \\ (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= 17\mathbf{i} + 38\mathbf{j} - 31\mathbf{k}\end{aligned}}$$

Because

$$\begin{aligned}\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ &= (1 + 6 + 6)(2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) - (2 + 15 - 2)(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= 13(2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) - 15(\mathbf{i} + 2\mathbf{j} + 3\mathbf{k}) \\ &= 11\mathbf{i} + 35\mathbf{j} - 58\mathbf{k}\end{aligned}$$

and

$$\begin{aligned}(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \\ &= (1 + 6 + 6)(2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) - (2 + 10 - 3)(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) \\ &= 13(2\mathbf{i} + 5\mathbf{j} - \mathbf{k}) - 9(\mathbf{i} + 3\mathbf{j} + 2\mathbf{k}) = 17\mathbf{i} + 38\mathbf{j} - 31\mathbf{k}\end{aligned}$$

These two results clearly confirm that

$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) \neq (\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$ *so beware!*

Before we proceed, note the following concerning the unit vectors.

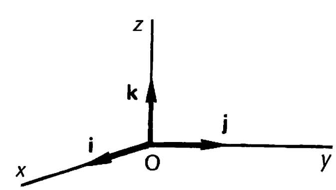
(a) $(\mathbf{i} \times \mathbf{j}) = \mathbf{k}$

$\therefore\ \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = \mathbf{i} \times \mathbf{k} = -\mathbf{j}$

$\therefore\ \mathbf{i} \times (\mathbf{i} \times \mathbf{j}) = -\mathbf{j}$

(b) $(\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = (0) \times \mathbf{j} = 0$

$\therefore\ (\mathbf{i} \times \mathbf{i}) \times \mathbf{j} = 0$



and once again, we see that

$\mathbf{i} \times (\mathbf{i} \times \mathbf{j}) \neq (\mathbf{i} \times \mathbf{i}) \times \mathbf{j}$

On to the next

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Finally, by way of revision:

Example 4

If $\mathbf{A} = 5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{C} = \mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$; determine

(a) the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

(b) the vector triple products (1) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$
(2) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

Finish all these and then check with the next frame

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(a) $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = -12$
(b) (1) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = 62\mathbf{i} + 44\mathbf{j} - 74\mathbf{k}$
(2) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C} = 109\mathbf{i} + 7\mathbf{j} - 22\mathbf{k}$

Here is the working.

$$\text{(a) } \mathbf{A} \cdot (\mathbf{B} \times \mathbf{C}) = \begin{vmatrix} 5 & -2 & 3 \\ 3 & 1 & -2 \\ 1 & -3 & 4 \end{vmatrix}$$

$$= 5(4 - 6) + 2(12 + 2) + 3(-9 - 1) = -12$$

$$\begin{aligned} \text{(b) (1) } \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) &= (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C} \\ &= (5 + 6 + 12)(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) \\ &\qquad - (15 - 2 - 6)(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \\ &= 23(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - 7(\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}) \\ &= 62\mathbf{i} + 44\mathbf{j} - 74\mathbf{k} \end{aligned}$$

$$\begin{aligned} \text{(2) } (\mathbf{A} \times \mathbf{B}) \times \mathbf{C} &= (\mathbf{C} \cdot \mathbf{A})\mathbf{B} - (\mathbf{C} \cdot \mathbf{B})\mathbf{A} \\ &= 23(3\mathbf{i} + \mathbf{j} - 2\mathbf{k}) - (-8)(5\mathbf{i} - 2\mathbf{j} + 3\mathbf{k}) \\ &= 109\mathbf{i} + 7\mathbf{j} - 22\mathbf{k} \end{aligned}$$

Let us now move to the next topic

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Differentiation of vectors

In many practical problems, we often deal with vectors that change with time, e.g. velocity, acceleration, etc. If a vector $\mathbf{A}$ depends on a scalar variable t, then $\mathbf{A}$ can be represented as $\mathbf{A}(t)$ and $\mathbf{A}$ is then said to be a function of t.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ then a_x, a_y, a_z will also be dependent on the parameter t.

i.e. $\mathbf{A}(t) = a_x(t)\mathbf{i} + a_y(t)\mathbf{j} + a_z(t)\mathbf{k}$

Differentiating with respect to t gives ....

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$$\boxed{\frac{\mathrm{d}}{\mathrm{d}t}\{\mathbf{A}(t)\} = \mathbf{i}\frac{\mathrm{d}}{\mathrm{d}t}\{a_x(t)\} + \mathbf{j}\frac{\mathrm{d}}{\mathrm{d}t}\{a_y(t)\} + \mathbf{k}\frac{\mathrm{d}}{\mathrm{d}t}\{a_z(t)\}}$$

In short $\dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}t} = \mathbf{i}\dfrac{\mathrm{d}a_x}{\mathrm{d}t} + \mathbf{j}\dfrac{\mathrm{d}a_y}{\mathrm{d}t} + \mathbf{k}\dfrac{\mathrm{d}a_z}{\mathrm{d}t}$.

The independent scalar variable is not, of course, restricted to t. In general, if u is the parameter, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u} = \ldots\ldots\ldots\ldots$$

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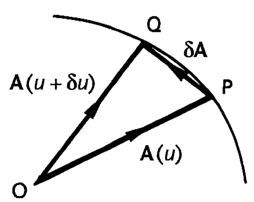
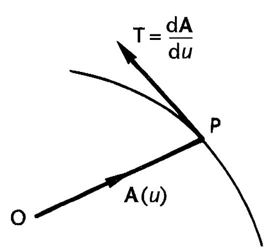
$$\boxed{\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u} = \mathbf{i}\frac{\mathrm{d}a_x}{\mathrm{d}u} + \mathbf{j}\frac{\mathrm{d}a_y}{\mathrm{d}u} + \mathbf{k}\frac{\mathrm{d}a_z}{\mathrm{d}u}}$$





If a position vector $\overline{\mathrm{OP}}$ moves to $\overline{\mathrm{OQ}}$ when u becomes $u + \delta u$, then as $\delta u \to 0$, the direction of the chord $\overline{\mathrm{PQ}}$ becomes that of the tangent to the curve at **P**, i.e. the direction of $\dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}$ is along the tangent to the locus of **P**.

Example 1

If $\mathbf{A} = (3u^2 + 4)\mathbf{i} + (2u - 5)\mathbf{j} + 4u^3\mathbf{k}$, then

$$\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}u} = 6u\mathbf{i} + 2\mathbf{j} + 12u^2\mathbf{k}}$$

If we differentiate this again, we get $\dfrac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}u^2} = 6\mathbf{i} + 24u\mathbf{k}$

When $u = 2$, $\dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}u} = 12\mathbf{i} + 2\mathbf{j} + 48\mathbf{k}$ and $\dfrac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}u^2} = 6\mathbf{i} + 48\mathbf{k}$

Then $\left|\dfrac{\mathrm{d}\mathbf{A}}{\mathrm{d}u}\right| = \ldots\ldots\ldots\ldots$ and $\left|\dfrac{\mathrm{d}^2\mathbf{A}}{\mathrm{d}u^2}\right| = \ldots\ldots\ldots\ldots$

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$$\left|\frac{d\mathbf{A}}{du}\right| = 49{\cdot}52; \quad \left|\frac{d^2\mathbf{A}}{du^2}\right| = 48{\cdot}37$$

Because

$$\left|\frac{d\mathbf{A}}{du}\right| = \{12^2 + 2^2 + 48^2\}^{1/2} = \{2452\}^{1/2} = 49{\cdot}52$$

and $$\left|\frac{d^2\mathbf{A}}{du^2}\right| = \{6^2 + 48^2\}^{1/2} = \{2340\}^{1/2} = 48{\cdot}37$$

Example 2

If $\mathbf{F} = \mathbf{i}\sin 2t + \mathbf{j}e^{3t} + \mathbf{k}(t^3 - 4t)$, then when $t = 1$

$$\frac{d\mathbf{F}}{dt} = \dots\dots\dots\dots; \quad \frac{d^2\mathbf{F}}{dt^2} = \dots\dots\dots\dots$$

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$$\frac{d\mathbf{F}}{dt} = 2\cos 2\mathbf{i} + 3e^3\mathbf{j} - \mathbf{k}$$
$$\frac{d^2\mathbf{F}}{dt^2} = -4\sin 2\mathbf{i} + 9e^3\mathbf{j} + 6\mathbf{k}$$

From these, we could if required find the magnitudes of $\frac{d\mathbf{F}}{dt}$ and $\frac{d^2\mathbf{F}}{dt^2}$.

$$\left|\frac{d\mathbf{F}}{dt}\right| = \dots\dots\dots\dots; \quad \left|\frac{d^2\mathbf{F}}{dt^2}\right| = \dots\dots\dots\dots$$

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$$\left|\frac{d\mathbf{F}}{dt}\right| = 60{\cdot}27; \quad \left|\frac{d^2\mathbf{F}}{dt^2}\right| = 180{\cdot}9$$

Because

$$\left|\frac{d\mathbf{F}}{dt}\right| = \{(2\cos 2)^2 + 9e^6 + 1\}^{1/2}$$
$$= \{0{\cdot}6927 + 3631 + 1\}^{1/2} = 60{\cdot}27$$

and $$\left|\frac{d^2\mathbf{F}}{dt^2}\right| = \{(-4\sin 2)^2 + 81e^6 + 36\}^{1/2}$$
$$= \{13{\cdot}23 + 32\,678 + 36\}^{1/2} = 180{\cdot}9$$

One more example.

Example 3

If $\mathbf{A} = (u+3)\mathbf{i} - (2+u^2)\mathbf{j} + 2u^3\mathbf{k}$, determine

(a) $\frac{d\mathbf{A}}{du}$ (b) $\frac{d^2\mathbf{A}}{du^2}$ (c) $\left|\frac{d\mathbf{A}}{du}\right|$ (d) $\left|\frac{d^2\mathbf{A}}{du^2}\right|$ at $u = 3$.

Work through all sections and then check with the next frame

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Here is the working. $\mathbf{A} = (u+3)\mathbf{i} - (2+u^2)\mathbf{j} + 2u^3\mathbf{k}$

(a) $\dfrac{d\mathbf{A}}{du} = \mathbf{i} - 2u\mathbf{j} + 6u^2\mathbf{k}$ At $u = 3$, $\dfrac{d\mathbf{A}}{du} = \mathbf{i} - 6\mathbf{j} + 54\mathbf{k}$

(b) $\dfrac{d^2\mathbf{A}}{du^2} = -2\mathbf{j} + 12u\mathbf{k}$ At $u = 3$, $\dfrac{d^2\mathbf{A}}{du^2} = -2\mathbf{j} + 36\mathbf{k}$

(c) $\left|\dfrac{d\mathbf{A}}{du}\right| = \{1 + 36 + 2916\}^{1/2} = (2953)^{1/2} = 54{\cdot}34$

(d) $\left|\dfrac{d^2\mathbf{A}}{du^2}\right| = \{4 + 1296\}^{1/2} = (1300)^{1/2} = 36{\cdot}06$

The next example is of a rather different kind, so move on

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Example 4

A particle moves in space so that at time t its position is stated as $x = 2t + 3$, $y = t^2 + 3t$, $z = t^3 + 2t^2$. We are required to find the components of its velocity and acceleration in the direction of the vector $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ when $t = 1$.

First we can write the position as a vector $\mathbf{r}$

$$\mathbf{r} = (2t+3)\mathbf{i} + (t^2 + 3t)\mathbf{j} + (t^3 + 2t^2)\mathbf{k}$$

Then, at $t = 1$

$$\frac{d\mathbf{r}}{dt} = \ldots\ldots\ldots\ldots; \qquad \frac{d^2\mathbf{r}}{dt^2} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{d\mathbf{r}}{dt} = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}; \qquad \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j} + 10\mathbf{k}}$$

Because

$$\frac{d\mathbf{r}}{dt} = 2\mathbf{i} + (2t+3)\mathbf{j} + (3t^2 + 4t)\mathbf{k}$$

$$\therefore \text{ At } t = 1, \quad \frac{d\mathbf{r}}{dt} = 2\mathbf{i} + 5\mathbf{j} + 7\mathbf{k}$$

and $$\frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j} + (6t + 4)\mathbf{k}$$

$$\therefore \text{ At } t = 1, \quad \frac{d^2\mathbf{r}}{dt^2} = 2\mathbf{j} + 10\mathbf{k}$$

Now, a unit vector parallel to $2\mathbf{i} + 3\mathbf{j} + 4\mathbf{k}$ is $\ldots\ldots\ldots\ldots$

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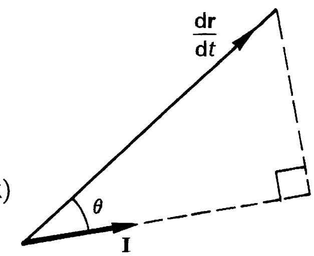
$$\boxed{\frac{2\mathbf{i}+3\mathbf{j}+4\mathbf{k}}{\sqrt{4+9+16}} = \frac{1}{\sqrt{29}}(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})}$$

Denote this unit vector by **I**. Then the component of $\dfrac{d\mathbf{r}}{dt}$ in the direction of **I**

$$= \frac{d\mathbf{r}}{dt}\cos\theta$$
$$= \frac{d\mathbf{r}}{dt}\cdot\mathbf{I}$$
$$= \frac{1}{\sqrt{29}}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$$
$$= \dots\dots\dots\dots$$



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$$\boxed{8{\cdot}73}$$

Because

$$\frac{1}{\sqrt{29}}(2\mathbf{i}+5\mathbf{j}+7\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}+4\mathbf{k}) = \frac{1}{\sqrt{29}}(4+15+28)$$
$$= \frac{47}{\sqrt{29}}$$
$$= 8{\cdot}73$$

Similarly, the component of $\dfrac{d^2\mathbf{r}}{dt^2}$ in the direction of **I** is

$\dots\dots\dots\dots$

40

$$\boxed{8{\cdot}54}$$

Because

$$\frac{d^2\mathbf{r}}{dt^2}\cos\theta = \frac{d^2\mathbf{r}}{dt^2}\cdot\mathbf{I}$$
$$= \frac{1}{\sqrt{29}}(2\mathbf{j}+10\mathbf{k})\cdot(2\mathbf{i}+3\mathbf{j}+4\mathbf{k})$$
$$= \frac{1}{\sqrt{29}}(6+40)$$
$$= \frac{46}{\sqrt{29}}$$
$$= 8{\cdot}54$$

▶

Differentiation of sums and products of vectors

If $\mathbf{A} = \mathbf{A}(u)$ and $\mathbf{B} = \mathbf{B}(u)$, then

(a) $\dfrac{d}{du}\{c\mathbf{A}\} = c\dfrac{d\mathbf{A}}{du}$

(b) $\dfrac{d}{du}\{\mathbf{A}+\mathbf{B}\} = \dfrac{d\mathbf{A}}{du} + \dfrac{d\mathbf{B}}{du}$

(c) $\dfrac{d}{du}\{\mathbf{A}\cdot\mathbf{B}\} = \mathbf{A}\cdot\dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du}\cdot\mathbf{B}$

(d) $\dfrac{d}{du}\{\mathbf{A}\times\mathbf{B}\} = \mathbf{A}\times\dfrac{d\mathbf{B}}{du} + \dfrac{d\mathbf{A}}{du}\times\mathbf{B}.$

These are very much like the normal rules of differentiation.
However, if $\mathbf{A}(u)\cdot\mathbf{A}(u) = a_x^2 + a_y^2 + a_z^2 = |\mathbf{A}|^2 = A^2$ is a constant then

$$\begin{aligned}\frac{d}{du}\{\mathbf{A}(u)\cdot\mathbf{A}(u)\} &= \mathbf{A}(u)\cdot\frac{d}{du}\{\mathbf{A}(u)\} + \mathbf{A}(u)\cdot\frac{d}{du}\{\mathbf{A}(u)\}\\ &= 2\mathbf{A}(u)\cdot\frac{d}{du}\{\mathbf{A}(u)\} = \frac{d}{du}\{\mathbf{A}^2\} = 0\end{aligned}$$

Assuming that $\mathbf{A}(u) \neq 0$, then since $\mathbf{A}(u)\cdot\dfrac{d}{du}\{\mathbf{A}(u)\} = 0$ it follows that $\mathbf{A}(u)$ and $\dfrac{d}{du}\{\mathbf{A}(u)\}$ are perpendicular vectors because

....

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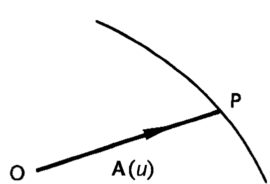
$$\mathbf{A}(u)\cdot\frac{d}{du}\{\mathbf{A}(u)\} = |\,\mathbf{A}(u)\,|\left|\frac{d}{du}\{\mathbf{A}(u)\}\right|\cos\theta = 0$$
$$\therefore\ \cos\theta = 0 \qquad \therefore\ \theta = \frac{\pi}{2}$$

Now let us deal with unit tangent vectors.

Unit tangent vectors

We have already established in Frame 30 of this Programme that if $\overline{OP}$ is a position vector $\mathbf{A}(u)$ in space, then the direction of the vector denoting $\dfrac{d}{du}\{\mathbf{A}(u)\}$ is



....

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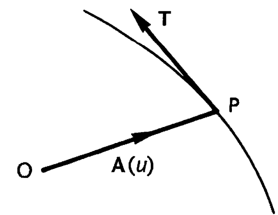
$\boxed{\text{parallel to the tangent to the curve at P}}$

Then the unit tangent vector **T** at P can be found from



$$\mathbf{T} = \frac{\dfrac{\mathrm{d}}{\mathrm{d}u}\{\mathbf{A}(u)\}}{\left|\dfrac{\mathrm{d}}{\mathrm{d}u}\{\mathbf{A}(u)\}\right|}$$

In simpler notation, this becomes:

If $\mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$ then the unit tangent vector **T** is given by

$$\mathbf{T} = \frac{\mathrm{d}\mathbf{r}/\mathrm{d}u}{|\mathrm{d}\mathbf{r}/\mathrm{d}u|}$$

Example 1

Determine the unit tangent vector at the point (2, 4, 7) for the curve with parametric equations $x = 2u$; $y = u^2 + 3$; $z = 2u^2 + 5$.

First we see that the point (2, 4, 7) corresponds to $u = 1$.

The vector equation of the curve is

$$\mathbf{r} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = 2u\mathbf{i} + (u^2 + 3)\mathbf{j} + (2u^2 + 5)\mathbf{k}$$

$$\therefore \quad \frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = \ldots\ldots\ldots\ldots$$

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$$\boxed{\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = 2\mathbf{i} + 2u\mathbf{j} + 4u\mathbf{k}}$$

and at $u = 1$, $\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}u} = 2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}$

Hence $\left|\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| = \ldots\ldots\ldots\ldots$ and $\mathbf{T} = \ldots\ldots\ldots\ldots$

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$$\boxed{\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| = 2\sqrt{6}; \quad \mathbf{T} = \frac{1}{\sqrt{6}}\{\mathbf{i} + \mathbf{j} + 2\mathbf{k}\}}$$

Because

$$\left|\frac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right| = \{4 + 4 + 16\}^{1/2} = 24^{1/2} = 2\sqrt{6}$$

$$\mathbf{T} = \frac{\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}}{\left|\dfrac{\mathrm{d}\mathbf{r}}{\mathrm{d}u}\right|} = \frac{2\mathbf{i} + 2\mathbf{j} + 4\mathbf{k}}{2\sqrt{6}} = \frac{1}{\sqrt{6}}\{\mathbf{i} + \mathbf{j} + 2\mathbf{k}\}$$

▶

Let us do another.

Example 2

Find the unit tangent vector at the point $(2, 0, \pi)$ for the curve with parametric equations $x = 2\sin\theta$; $y = 3\cos\theta$; $z = 2\theta$.

We see that the point $(2, 0, \pi)$ corresponds to $\theta = \pi/2$.

Writing the curve in vector form $\mathbf{r} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{r} = 2\sin\theta\,\mathbf{i} + 3\cos\theta\,\mathbf{j} + 2\theta\,\mathbf{k}}$$

Then, at $\theta = \pi/2$, $\dfrac{d\mathbf{r}}{d\theta} = \ldots\ldots\ldots\ldots$

$$\left|\frac{d\mathbf{r}}{d\theta}\right| = \ldots\ldots\ldots\ldots$$

$$\mathbf{T} = \ldots\ldots\ldots\ldots$$

Finish it off

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$$\boxed{\begin{array}{l}\dfrac{d\mathbf{r}}{d\theta} = -3\mathbf{j} + 2\mathbf{k}; \quad \left|\dfrac{d\mathbf{r}}{d\theta}\right| = \sqrt{13} \\ \mathbf{T} = \dfrac{1}{\sqrt{13}}(-3\mathbf{j} + 2\mathbf{k})\end{array}}$$

And now

Example 3

Determine the unit tangent vector for the curve

$x = 3t$; $\quad y = 2t^2$; $\quad z = t^2 + t$

at the point (6, 8, 6).

On your own. $\mathbf{T} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{T} = \frac{1}{\sqrt{98}}(3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k})}$$

The point (6, 8, 6) corresponds to $t = 2$

$$\mathbf{r} = 3t\mathbf{i} + 2t^2\mathbf{j} + (t^2 + t)\mathbf{k}$$

$$\therefore \quad \frac{d\mathbf{r}}{dt} = 3\mathbf{i} + 4t\mathbf{j} + (2t + 1)\mathbf{k}$$

At $t = 2$, $\mathbf{r} = 6\mathbf{i} + 8\mathbf{j} + 6\mathbf{k}$ and $\dfrac{d\mathbf{r}}{dt} = 3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k}$

$$\therefore \quad \left|\frac{d\mathbf{r}}{dt}\right| = (9 + 64 + 25)^{1/2} = \sqrt{98}$$

$$\therefore \quad \mathbf{T} = \frac{d\mathbf{r}/dt}{|d\mathbf{r}/dt|} = \frac{1}{\sqrt{98}}(3\mathbf{i} + 8\mathbf{j} + 5\mathbf{k})$$

Partial differentiation of vectors

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If a vector $\mathbf{F}$ is a function of two independent variables u and v, then the rules of differentiation follow the usual pattern.
If $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ then x, y, z will also be functions of u and v.

Then
$$\frac{\partial \mathbf{F}}{\partial u} = \frac{\partial x}{\partial u}\mathbf{i} + \frac{\partial y}{\partial u}\mathbf{j} + \frac{\partial z}{\partial u}\mathbf{k}$$
$$\frac{\partial \mathbf{F}}{\partial v} = \frac{\partial x}{\partial v}\mathbf{i} + \frac{\partial y}{\partial v}\mathbf{j} + \frac{\partial z}{\partial v}\mathbf{k}$$
$$\frac{\partial^2 \mathbf{F}}{\partial u^2} = \frac{\partial^2 x}{\partial u^2}\mathbf{i} + \frac{\partial^2 y}{\partial u^2}\mathbf{j} + \frac{\partial^2 z}{\partial u^2}\mathbf{k}$$
$$\frac{\partial^2 \mathbf{F}}{\partial v^2} = \frac{\partial^2 x}{\partial v^2}\mathbf{i} + \frac{\partial^2 y}{\partial v^2}\mathbf{j} + \frac{\partial^2 z}{\partial v^2}\mathbf{k}$$
$$\frac{\partial^2 \mathbf{F}}{\partial u \partial v} = \frac{\partial^2 x}{\partial u \partial v}\mathbf{i} + \frac{\partial^2 y}{\partial u \partial v}\mathbf{j} + \frac{\partial^2 z}{\partial u \partial v}\mathbf{k}$$

and for small finite changes $\mathrm{d}u$ and $\mathrm{d}v$ in u and v, we have

$$\mathrm{d}\mathbf{F} = \frac{\partial \mathbf{F}}{\partial u}\mathrm{d}u + \frac{\partial \mathbf{F}}{\partial v}\mathrm{d}v$$

Example

If $\mathbf{F} = 2uv\mathbf{i} + (u^2 - 2v)\mathbf{j} + (u + v^2)\mathbf{k}$

$$\frac{\partial \mathbf{F}}{\partial u} = \dots\dots\dots\dots; \quad \frac{\partial \mathbf{F}}{\partial v} = \dots\dots\dots\dots$$
$$\frac{\partial^2 \mathbf{F}}{\partial u^2} = \dots\dots\dots\dots; \quad \frac{\partial^2 \mathbf{F}}{\partial u \partial v} = \dots\dots\dots\dots$$

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$$\boxed{\begin{array}{ll} \dfrac{\partial \mathbf{F}}{\partial u} = 2v\mathbf{i} + 2u\mathbf{j} + \mathbf{k}; & \dfrac{\partial \mathbf{F}}{\partial v} = 2u\mathbf{i} - 2\mathbf{j} + 2v\mathbf{k} \\ \dfrac{\partial^2 \mathbf{F}}{\partial u^2} = 2\mathbf{j}; & \dfrac{\partial^2 \mathbf{F}}{\partial u \partial v} = 2\mathbf{i} \end{array}}$$

This is straightforward enough.

Integration of vector functions

The process is the reverse of that for differentiation. If a vector $\mathbf{F} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ where $\mathbf{F}$, x, y, z are expressed as functions of u, then

$$\int_a^b \mathbf{F}\,\mathrm{d}u = \mathbf{i}\int_a^b x\,\mathrm{d}u + \mathbf{j}\int_a^b y\,\mathrm{d}u + \mathbf{k}\int_a^b z\,\mathrm{d}u.$$

Example 1

If $\mathbf{F} = (3t^2 + 4t)\mathbf{i} + (2t - 5)\mathbf{j} + 4t^3\mathbf{k}$, then

$$\int_1^3 \mathbf{F}\,\mathrm{d}t = \mathbf{i}\int_1^3 (3t^2 + 4t)\,\mathrm{d}t + \mathbf{j}\int_1^3 (2t - 5)\,\mathrm{d}t + \mathbf{k}\int_1^3 4t^3\,\mathrm{d}t = \dots\dots\dots\dots$$

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$$\boxed{42\mathbf{i} - 2\mathbf{j} + 80\mathbf{k}}$$

Because

$$\int_1^3 \mathbf{F}\,dt = \left[\mathbf{i}(t^3 + 2t^2) + \mathbf{j}(t^2 - 5t) + \mathbf{k}t^4\right]_1^3$$
$$= (45\mathbf{i} - 6\mathbf{j} + 81\mathbf{k}) - (3\mathbf{i} - 4\mathbf{j} + \mathbf{k}) = 42\mathbf{i} - 2\mathbf{j} + 80\mathbf{k}$$

Here is a slightly different one.

Example 2

If $\quad \mathbf{F} = 3u\mathbf{i} + u^2\mathbf{j} + (u+2)\mathbf{k}$

and $\quad \mathbf{V} = 2u\mathbf{i} - 3u\mathbf{j} + (u-2)\mathbf{k}$

evaluate $\displaystyle\int_0^2 (\mathbf{F} \times \mathbf{V})\,du$.

First we must determine $\mathbf{F} \times \mathbf{V}$ in terms of u.

$$\mathbf{F} \times \mathbf{V} = \dots\dots\dots\dots$$

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$$\boxed{\mathbf{F} \times \mathbf{V} = (u^3 + u^2 + 6u)\mathbf{i} - (u^2 - 10u)\mathbf{j} - (2u^3 + 9u^2)\mathbf{k}}$$

Because

$$\mathbf{F} \times \mathbf{V} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3u & u^2 & (u+2) \\ 2u & -3u & (u-2) \end{vmatrix}$$

which gives the result above.

Then $\displaystyle\int_0^2 (\mathbf{F} \times \mathbf{V})\,du = \dots\dots\dots\dots$

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$$\boxed{\tfrac{4}{3}\{14\mathbf{i} + 13\mathbf{j} - 24\mathbf{k}\}}$$

Because

$$\int (\mathbf{F} \times \mathbf{V})\,du = \left(\frac{u^4}{4} + \frac{u^3}{3} + 3u^2\right)\mathbf{i} - \left(\frac{u^3}{3} - 5u^2\right)\mathbf{j} - \left(\frac{u^4}{2} + 3u^3\right)\mathbf{k}$$

$$\therefore \int_0^2 (\mathbf{F} \times \mathbf{V})\,du = (4 + \tfrac{8}{3} + 12)\mathbf{i} - (\tfrac{8}{3} - 20)\mathbf{j} - (8 + 24)\mathbf{k}$$
$$= \tfrac{4}{3}\{14\mathbf{i} + 13\mathbf{j} - 24\mathbf{k}\}$$

▶

Example 3

If $\mathbf{F} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$ where

$$\mathbf{A} = 3t^2\mathbf{i} + (2t - 3)\mathbf{j} + 4t\mathbf{k}$$
$$\mathbf{B} = 2\mathbf{i} + 4t\mathbf{j} + 3(1 - t)\mathbf{k}$$
$$\mathbf{C} = 2t\mathbf{i} - 3t^2\mathbf{j} - 2t\mathbf{k}$$

determine $\int_0^1 \mathbf{F}dt$.

First we need to find $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$. The simplest way to do this is to use the relationship

$$\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = \dots\dots\dots\dots$$

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$$\boxed{\mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (\mathbf{A} \cdot \mathbf{C})\mathbf{B} - (\mathbf{A} \cdot \mathbf{B})\mathbf{C}}$$

So $\quad \mathbf{A} \cdot \mathbf{C} = \dots\dots\dots\dots$

and $\quad \mathbf{A} \cdot \mathbf{B} = \dots\dots\dots\dots$

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$$\boxed{\begin{aligned} &\mathbf{A} \cdot \mathbf{C} = 6t^3 - 6t^3 + 9t^2 - 8t^2 = t^2 \\ &\mathbf{A} \cdot \mathbf{B} = 6t^2 + 8t^2 - 12t + 12t - 12t^2 = 2t^2 \end{aligned}}$$

Then $\mathbf{F} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C})$

$$= t^2\{2\mathbf{i} + 4t\mathbf{j} + 3(1 - t)\mathbf{k}\} - 2t^2\{2t\mathbf{i} - 3t^2\mathbf{j} - 2t\mathbf{k}\}$$

$$\therefore \int_0^1 \mathbf{F}\,dt = \dots\dots\dots\dots$$

Finish off the simplification and complete the integration.

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$$\boxed{\tfrac{1}{60}\{-20\mathbf{i} + 132\mathbf{j} + 75\mathbf{k}\}}$$

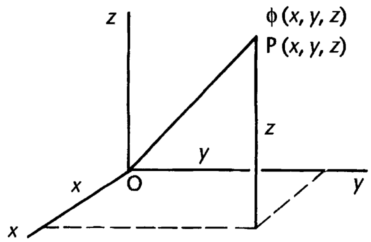
Because

$$\mathbf{F} = \mathbf{A} \times (\mathbf{B} \times \mathbf{C}) = (2t^2 - 4t^3)\mathbf{i} + (4t^3 + 6t^4)\mathbf{j} + (3t^2 + t^3)\mathbf{k}$$

Integration with respect to t then gives the result stated above.

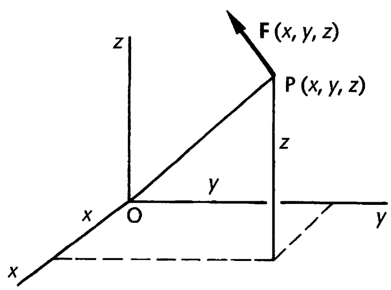
Now let us move on to the next stage of our development

Scalar and vector fields



If every point P (x, y, z) of a region R of space has associated with it a scalar quantity $\phi(x, y, z)$, then $\phi(x, y, z)$ is a *scalar function* and a *scalar field* is said to exist in the region R. 56

Examples of scalar fields are temperature, potential, etc.



Similarly, if every point P (x, y, z) of a region R has associated with it a vector quantity $\mathbf{F}(x, y, z)$, then $\mathbf{F}(x, y, z)$ is a *vector function* and a *vector field* is said to exist in the region R.

Examples of vector fields are force, velocity, acceleration, etc. $\mathbf{F}(x, y, z)$ can be defined in terms of its components parallel to the coordinate axes, OX, OY, OZ.

That is, $\mathbf{F}(x, y, z) = F_x\mathbf{i} + F_y\mathbf{j} + F_z\mathbf{k}$.

Note these important definitions:
we shall be making good use of them as we proceed

Grad (gradient of a scalar function) 57

If a scalar function $\phi(x, y, z)$ is continuously differentiable with respect to its variables x, y, z, throughout the region, then the *gradient* of ϕ, written *grad* ϕ, is defined as the vector

$$\text{grad}\,\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k} \qquad (12)$$

Note that, while ϕ is a scalar function, grad ϕ is a vector function. For example, if ϕ depends upon the position of P and is defined by $\phi = 2x^2yz^3$, then

$$\text{grad}\,\phi = 4xyz^3\mathbf{i} + 2x^2z^3\mathbf{j} + 6x^2yz^2\mathbf{k}$$

▶

Notation

The expression (12) above can be written

$$\text{grad}\,\phi = \left\{\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right\}\phi$$

where $\left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$ is called a *vector differential operator* and is denoted by the symbol ∇ (pronounced 'del' or sometimes 'nabla')

i.e. $$\nabla \equiv \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

Beware! ∇ cannot exist alone: it is an operator and must operate on a stated scalar function $\phi(x, y, z)$.

If $\mathbf{F}$ is a vector function, $\nabla\mathbf{F}$ has no meaning.

So we have:

$$\begin{aligned}\nabla\phi = \text{grad}\,\phi &= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\phi \\ &= \mathbf{i}\frac{\partial\phi}{\partial x} + \mathbf{j}\frac{\partial\phi}{\partial y} + \mathbf{k}\frac{\partial\phi}{\partial z}\end{aligned} \tag{13}$$

Make a note of this definition and then let us see how to use it

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Example 1

If $\phi = x^2yz^3 + xy^2z^2$, determine grad ϕ at the point P (1, 3, 2).

By the definition, $\quad \text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}.$

All we have to do then is to find the partial derivatives at $x = 1$, $y = 3$, $z = 2$ and insert their values.

$$\therefore\ \nabla\phi = \dots\dots\dots\dots$$

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$$\boxed{4(21\mathbf{i} + 8\mathbf{j} + 18\mathbf{k})}$$

Because

$$\phi = x^2yz^3 + xy^2z^2 \qquad \therefore\ \frac{\partial\phi}{\partial x} = 2xyz^3 + y^2z^2$$

$$\frac{\partial\phi}{\partial y} = x^2z^3 + 2xyz^2 \qquad \frac{\partial\phi}{\partial z} = 3x^2yz^2 + 2xy^2z$$

Then, at (1, 3, 2)

$$\frac{\partial\phi}{\partial x} = 48 + 36 \qquad \therefore\ \frac{\partial\phi}{\partial x} = 84$$

$$\frac{\partial\phi}{\partial y} = 8 + 24 \qquad \therefore\ \frac{\partial\phi}{\partial y} = 32$$

$$\frac{\partial\phi}{\partial z} = 36 + 36 \qquad \therefore\ \frac{\partial\phi}{\partial z} = 72$$

$$\therefore\ \text{grad}\,\phi = \nabla\phi = 84\mathbf{i} + 32\mathbf{j} + 72\mathbf{k} = 4(21\mathbf{i} + 8\mathbf{j} + 18\mathbf{k})$$

Example 2

If $\mathbf{A} = x^2z\mathbf{i} + xy\mathbf{j} + y^2z\mathbf{k}$
and $\mathbf{B} = yz^2\mathbf{i} + xz\mathbf{j} + x^2z\mathbf{k}$

determine an expression for grad $(\mathbf{A} \cdot \mathbf{B})$.

This we can soon do since we know that $\mathbf{A} \cdot \mathbf{B}$ is a scalar function of x, y and z.

First then, $\mathbf{A} \cdot \mathbf{B} = \ldots\ldots\ldots\ldots$

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$$\boxed{\mathbf{A} \cdot \mathbf{B} = x^2yz^3 + x^2yz + x^2y^2z^2}$$

Then $\nabla(\mathbf{A} \cdot \mathbf{B}) = \ldots\ldots\ldots\ldots$

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$$\boxed{2xyz(z^2 + 1 + yz)\mathbf{i} + x^2z(z^2 + 1 + 2yz)\mathbf{j} + x^2y(3z^2 + 1 + 2yz)\mathbf{k}}$$

Because

$$\text{if } \phi = \mathbf{A} \cdot \mathbf{B} = (x^2z\mathbf{i} + xy\mathbf{j} + y^2z\mathbf{k}) \cdot (yz^2\mathbf{i} + xz\mathbf{j} + x^2z\mathbf{k})$$

$$= x^2yz^3 + x^2yz + x^2y^2z^2$$

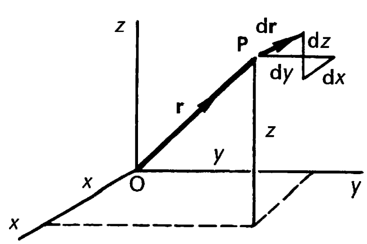
$$\frac{\partial \phi}{\partial x} = 2xyz^3 + 2xyz + 2xy^2z^2 = 2xyz(z^2 + 1 + yz)$$

$$\frac{\partial \phi}{\partial y} = x^2z^3 + x^2z + 2x^2yz^2 = x^2z(z^2 + 1 + 2yz)$$

$$\frac{\partial \phi}{\partial z} = 3x^2yz^2 + x^2y + 2x^2y^2z = x^2y(3z^2 + 1 + 2yz)$$

$$\therefore\ \nabla(\mathbf{A} \cdot \mathbf{B}) = 2xyz(z^2 + 1 + yz)\mathbf{i} + x^2z(z^2 + 1 + 2yz)\mathbf{j}$$
$$+ x^2y(3z^2 + 1 + 2yz)\mathbf{k}$$

Now let us obtain another useful relationship.



If $\overline{\text{OP}}$ is a position vector $\mathbf{r}$ where $\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$ and $\mathrm{d}\mathbf{r}$ is a small displacement corresponding to changes $\mathrm{d}x$, $\mathrm{d}y$, $\mathrm{d}z$ in x, y, z respectively, then

$$\mathrm{d}\mathbf{r} = \mathrm{d}x\,\mathbf{i} + \mathrm{d}y\,\mathbf{j} + \mathrm{d}z\,\mathbf{k}$$

If $\phi(x, y, z)$ is a scalar function at P, we know that

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

Then $\text{grad}\,\phi \cdot \mathrm{d}\mathbf{r} = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{grad}\,\phi\cdot\mathbf{dr}=\frac{\partial\phi}{\partial x}\text{d}x+\frac{\partial\phi}{\partial y}\text{d}y+\frac{\partial\phi}{\partial z}\text{d}z}$$

Because

$$\begin{aligned}\text{grad}\,\phi\cdot\mathbf{dr}&=\left(\frac{\partial\phi}{\partial x}\mathbf{i}+\frac{\partial\phi}{\partial y}\mathbf{j}+\frac{\partial\phi}{\partial z}\mathbf{k}\right)\cdot(\text{d}x\,\mathbf{i}+\text{d}y\,\mathbf{j}+\text{d}z\,\mathbf{k})\\&=\frac{\partial\phi}{\partial x}\text{d}x+\frac{\partial\phi}{\partial y}\text{d}y+\frac{\partial\phi}{\partial z}\text{d}z\\&=\text{the total differential d}\phi\text{ of }\phi\end{aligned}$$

That is

$$\text{d}\phi=\mathbf{dr}\cdot\text{grad}\,\phi \qquad (14)$$

This will certainly be useful, so make a note of it

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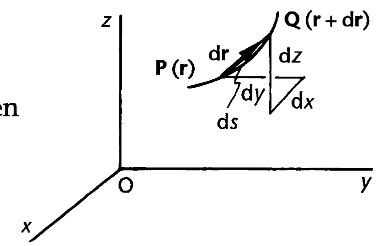
Directional derivatives

We have just established that

$$\text{d}\phi=\mathbf{dr}\cdot\text{grad}\,\phi$$



If ds is the small element of arc between P ($\mathbf{r}$) and Q ($\mathbf{r}+\mathbf{dr}$) then d$s=|\mathbf{dr}|$

$$\frac{\mathbf{dr}}{\text{d}s}=\frac{\mathbf{dr}}{|\mathbf{dr}|}$$

and $\dfrac{\mathbf{dr}}{\text{d}s}$ is thus a unit vector in the direction of $\mathbf{dr}$.

$$\therefore\ \frac{\text{d}\phi}{\text{d}s}=\frac{\mathbf{dr}}{\text{d}s}\cdot\text{grad}\,\phi$$

If we denote the unit vector $\dfrac{\mathbf{dr}}{\text{d}s}$ by $\hat{\mathbf{a}}$ then the result becomes

$$\frac{\text{d}\phi}{\text{d}s}=\hat{\mathbf{a}}\cdot\text{grad}\,\phi$$

$\dfrac{\text{d}\phi}{\text{d}s}$ is thus the projection of grad ϕ on the unit vector $\hat{\mathbf{a}}$ and is called the *directional derivative* of ϕ in the direction of $\hat{\mathbf{a}}$. It gives the rate of change of ϕ with distance measured in the direction of $\hat{\mathbf{a}}$ and $\dfrac{\text{d}\phi}{\text{d}s}=\hat{\mathbf{a}}\cdot\text{grad}\,\phi$ will be a maximum when $\hat{\mathbf{a}}$ and grad ϕ have the same direction, since then

$\hat{\mathbf{a}}\cdot\text{grad}\,\phi=|\,\hat{\mathbf{a}}\,||\text{grad}\,\phi\,|\cos\theta$ and θ will be zero.

Thus the direction of grad ϕ gives the direction in which the maximum rate of change of ϕ occurs.

▶

Example 1

Find the directional derivative of the function $\phi = x^2z + 2xy^2 + yz^2$ at the point (1, 2, –1) in the direction of the vector $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$.

We start off with $\phi = x^2z + 2xy^2 + yz^2$

$$\therefore \ \nabla\phi = \ldots\ldots\ldots\ldots$$

64

$$\boxed{\nabla\phi = (2xz + 2y^2)\mathbf{i} + (4xy + z^2)\mathbf{j} + (x^2 + 2yz)\mathbf{k}}$$

Because

$$\frac{\partial\phi}{\partial x} = 2xz + 2y^2; \qquad \frac{\partial\phi}{\partial y} = 4xy + z^2; \qquad \frac{\partial\phi}{\partial z} = x^2 + 2yz$$

Then, at (1, 2, – 1)

$$\nabla\phi = (-2 + 8)\mathbf{i} + (8 + 1)\mathbf{j} + (1 - 4)\mathbf{k} = 6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$$

Next we have to find the unit vector $\hat{\mathbf{a}}$ where $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$

$$\hat{\mathbf{a}} = \ldots\ldots\ldots\ldots$$

65

$$\boxed{\hat{\mathbf{a}} = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})}$$

Because

$$\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k} \quad \therefore \quad |\,\mathbf{A}\,| = \sqrt{4 + 9 + 16} = \sqrt{29}$$

$$\hat{\mathbf{a}} = \frac{\mathbf{A}}{|\,\mathbf{A}\,|} = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$$

So we have $\nabla\phi = 6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k}$ and $\hat{\mathbf{a}} = \dfrac{1}{\sqrt{29}}\,(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k})$

$$\therefore \ \frac{\mathrm{d}\phi}{\mathrm{d}s} = \hat{\mathbf{a}} \cdot \nabla\phi$$

$$= \ldots\ldots\ldots\ldots$$

66

$$\boxed{\frac{d\phi}{ds} = \frac{51}{\sqrt{29}} = 9{\cdot}47}$$

Because

$$\frac{d\phi}{ds} = \hat{\mathbf{a}} \cdot \nabla\phi = \frac{1}{\sqrt{29}}(2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) \cdot (6\mathbf{i} + 9\mathbf{j} - 3\mathbf{k})$$

$$= \frac{1}{\sqrt{29}}(12 + 27 + 12) = \frac{51}{\sqrt{29}} = 9{\cdot}47$$

That is all there is to it.

(a) From the given scalar function ϕ, determine $\nabla\phi$.

(b) Find the unit vector $\hat{\mathbf{a}}$ in the direction of the given vector $\mathbf{A}$.

(c) Then $\dfrac{d\phi}{ds} = \hat{\mathbf{a}} \cdot \nabla\phi$.

Example 2

Find the directional derivative of $\phi = x^2y + y^2z + z^2x$ at the point $(1, -1, 2)$ in the direction of the vector $\mathbf{A} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}$.

Same as before. *Work through it and check the result with the next frame*

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$$\boxed{\frac{d\phi}{ds} = \frac{-23}{3\sqrt{5}} = -3{\cdot}43}$$

Because

$$\phi = x^2y + y^2z + z^2x$$

$$\therefore\ \nabla\phi = (2xy + z^2)\mathbf{i} + (x^2 + 2yz)\mathbf{j} + (y^2 + 2zx)\mathbf{k}$$

$$\therefore\ \text{At } (1, -1, 2),\quad \nabla\phi = 2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k}$$

$$\mathbf{A} = 4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k} \quad \therefore\ |\mathbf{A}| = \sqrt{16 + 4 + 25} = \sqrt{45} = 3\sqrt{5}$$

$$\therefore\ \hat{\mathbf{a}} = \frac{1}{3\sqrt{5}}(4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k})$$

$$\therefore\ \frac{d\phi}{ds} = \hat{\mathbf{a}} \cdot \nabla\phi = \frac{1}{3\sqrt{5}}(4\mathbf{i} + 2\mathbf{j} - 5\mathbf{k}) \cdot (2\mathbf{i} - 3\mathbf{j} + 5\mathbf{k})$$

$$= \frac{1}{3\sqrt{5}}(8 - 6 - 25) = \frac{-23}{3\sqrt{5}} = -3{\cdot}43$$

Example 3

Find the direction from the point $(1, 1, 0)$ which gives the greatest rate of increase of the function $\phi = (x + 3y)^2 + (2y - z)^2$.

This appears to be different, but it rests on the fact that the greatest rate of increase of ϕ with respect to distance is in

....

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$$\boxed{\text{the direction of } \nabla\phi}$$

All we need then is to find the vector $\nabla\phi$, which is

....

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$$\boxed{\nabla\phi = 4(2\mathbf{i} + 8\mathbf{j} - \mathbf{k})}$$

Because

$$\phi = (x + 3y)^2 + (2y - z)^2$$

$$\therefore \quad \frac{\partial\phi}{\partial x} = 2(x + 3y); \quad \frac{\partial\phi}{\partial y} = 6(x + 3y) + 4(2y - z); \quad \frac{\partial\phi}{\partial z} = -2(2y - z)$$

$$\therefore \text{ At } (1, 1, 0), \quad \frac{\partial\phi}{\partial x} = 8; \quad \frac{\partial\phi}{\partial y} = 32; \quad \frac{\partial\phi}{\partial z} = -4$$

$$\therefore \quad \nabla\phi = 8\mathbf{i} + 32\mathbf{j} - 4\mathbf{k} = 4(2\mathbf{i} + 8\mathbf{j} - \mathbf{k})$$

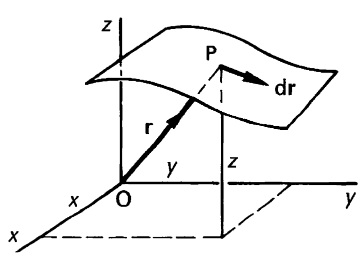
$\therefore$ greatest rate of increase occurs in direction $2\mathbf{i} + 8\mathbf{j} - \mathbf{k}$

So on we go

Unit normal vectors

70

The equation of $\phi(x, y, z) =$ constant represents a surface in space. For example, $3x - 4y + 2z = 1$ is the equation of a plane and $x^2 + y^2 + z^2 = 4$ represents a sphere centred on the origin and of radius 2.



If $\mathbf{dr}$ is a displacement in this surface, then $d\phi = 0$ since ϕ is constant over the surface.

Therefore our previous relationship $\mathbf{dr} \cdot \text{grad } \phi = d\phi$ becomes

$$\mathbf{dr} \cdot \text{grad } \phi = 0$$

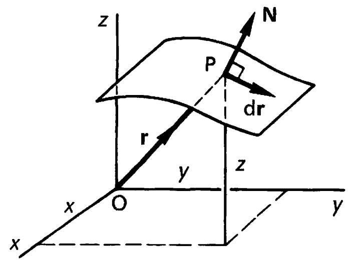
for all such small displacements $\mathbf{dr}$ in the surface.

But $\mathbf{dr} \cdot \text{grad}\,\phi = |\,\mathbf{dr}\,||\,\text{grad}\,\phi\,|\cos\theta = 0$.

$\therefore \ \theta = \dfrac{\pi}{2} \quad \therefore$ grad ϕ is perpendicular to $\mathbf{dr}$, i.e. grad ϕ is a vector perpendicular to the surface at P, in the direction of maximum rate of change of ϕ . The magnitude of that maximum rate of change is given by $|\text{ grad } \phi|$.

▶

The unit vector **N** in the direction of grad ϕ is called the *unit normal vector* at P.



$\therefore$ Unit normal vector

$$\mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|} \tag{15}$$

Example 1

Find the unit normal vector to the surface $x^3y + 4xz^2 + xy^2z + 2 = 0$ at the point $(1, 3, -1)$.

Vector normal $= \nabla\phi = \ldots\ldots\ldots\ldots$

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$$\boxed{\nabla\phi = (3x^2y + 4z^2 + y^2z)\mathbf{i} + (x^3 + 2xyz)\mathbf{j} + (8xz + xy^2)\mathbf{k}}$$

Then, at (1, 3, − 1), $\quad \nabla\phi = 4\mathbf{i} - 5\mathbf{j} + \mathbf{k}$

and the unit normal at (1, 3, − 1) is $\ldots\ldots\ldots\ldots$

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$$\boxed{\frac{1}{\sqrt{42}}(4\mathbf{i} - 5\mathbf{j} + \mathbf{k})}$$

Because

$$|\nabla\phi| = \sqrt{16 + 25 + 1} = \sqrt{42}$$

and $\quad \mathbf{N} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{1}{\sqrt{42}}(4\mathbf{i} - 5\mathbf{j} + \mathbf{k})$

One more.

Example 2

Determine the unit normal to the surface

$xyz + x^2y - 5yz - 5 = 0$ at the point (3, 1, 2).

All very straightforward. Complete it.

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$$\boxed{\text{Unit normal} = \mathbf{N} = \frac{1}{\sqrt{93}}(8\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})}$$

Because

$$\phi = xyz + x^2y - 5yz - 5$$

$$\therefore \ \nabla\phi = (yz + 2xy)\mathbf{i} + (xz + x^2 - 5z)\mathbf{j} + (xy - 5y)\mathbf{k}$$

At (3, 1, 2), $\nabla\phi = 8\mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$; $|\nabla\phi| = \sqrt{64 + 25 + 4} = \sqrt{93}$

$$\therefore \ \text{Unit normal} = \mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{\sqrt{93}}(8\mathbf{i} + 5\mathbf{j} - 2\mathbf{k})$$

Collecting our results so far, we have, for $\phi(x, y, z)$ a scalar function

(a) $\text{grad}\,\phi = \nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$

(b) $\mathrm{d}\phi = \mathbf{dr} \cdot \text{grad}\,\phi$ where $\mathrm{d}\phi = \dfrac{\partial\phi}{\partial x}\mathrm{d}x + \dfrac{\partial\phi}{\partial y}\mathrm{d}y + \dfrac{\partial\phi}{\partial z}\mathrm{d}z$

(c) directional derivative $\dfrac{\mathrm{d}\phi}{\mathrm{d}s} = \hat{\mathbf{a}} \cdot \text{grad}\,\phi$

(d) unit normal vector $\mathbf{N} = \dfrac{\nabla\phi}{|\nabla\phi|}$.

Copy out this brief summary for future reference. It will help

Grad of sums and products of scalars

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(a)
$$\nabla(A + B) = \mathbf{i}\left\{\frac{\partial}{\partial x}(A + B)\right\} + \mathbf{j}\left\{\frac{\partial}{\partial y}(A + B)\right\} + \mathbf{k}\left\{\frac{\partial}{\partial z}(A + B)\right\}$$
$$= \left\{\frac{\partial A}{\partial x}\mathbf{i} + \frac{\partial A}{\partial y}\mathbf{j} + \frac{\partial A}{\partial z}\mathbf{k}\right\} + \left\{\frac{\partial B}{\partial x}\mathbf{i} + \frac{\partial B}{\partial y}\mathbf{j} + \frac{\partial B}{\partial z}\mathbf{k}\right\}$$

$$\therefore \ \nabla(A + B) = \nabla A + \nabla B$$

(b)
$$\nabla(AB) = \mathbf{i}\left\{\frac{\partial}{\partial x}(AB)\right\} + \mathbf{j}\left\{\frac{\partial}{\partial y}(AB)\right\} + \mathbf{k}\left\{\frac{\partial}{\partial z}(AB)\right\}$$
$$= \mathbf{i}\left\{A\frac{\partial B}{\partial x} + B\frac{\partial A}{\partial x}\right\} + \mathbf{j}\left\{A\frac{\partial B}{\partial y} + B\frac{\partial A}{\partial y}\right\} + \mathbf{k}\left\{A\frac{\partial B}{\partial z} + B\frac{\partial A}{\partial z}\right\}$$
$$= \left\{A\frac{\partial B}{\partial x}\mathbf{i} + A\frac{\partial B}{\partial y}\mathbf{j} + A\frac{\partial B}{\partial z}\mathbf{k}\right\} + \left\{B\frac{\partial A}{\partial x}\mathbf{i} + B\frac{\partial A}{\partial y}\mathbf{j} + B\frac{\partial A}{\partial z}\mathbf{k}\right\}$$
$$= A\left\{\frac{\partial B}{\partial x}\mathbf{i} + \frac{\partial B}{\partial y}\mathbf{j} + \frac{\partial B}{\partial z}\mathbf{k}\right\} + B\left\{\frac{\partial A}{\partial x}\mathbf{i} + \frac{\partial A}{\partial y}\mathbf{j} + \frac{\partial A}{\partial z}\mathbf{k}\right\}$$

$$\therefore \ \nabla(AB) = A(\nabla B) + B(\nabla A)$$

Remember that in these results A and B are scalars. The operator ∇ acting on a vector …………

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has no meaning

Example

If $A = x^2yz + xz^2$ and $B = xy^2z - z^3$, evaluate $\nabla(AB)$ at the point (2, 1, 3).

We know that $\nabla(AB) = A(\nabla B) + B(\nabla A)$

At (2, 1, 3),

$$\nabla B = \ldots\ldots\ldots\ldots; \quad \nabla A = \ldots\ldots\ldots\ldots$$

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$$\boxed{\nabla B = 3\mathbf{i} + 12\mathbf{j} - 25\mathbf{k}; \quad \nabla A = 21\mathbf{i} + 12\mathbf{j} + 16\mathbf{k}}$$

$$\nabla B = \frac{\partial B}{\partial x}\mathbf{i} + \frac{\partial B}{\partial y}\mathbf{j} + \frac{\partial B}{\partial z}\mathbf{k} = y^2z\mathbf{i} + 2xyz\mathbf{j} + (xy^2 - 3z^2)\mathbf{k}$$

$$= 3\mathbf{i} + 12\mathbf{j} - 25\mathbf{k} \quad \text{at } (2, 1, 3)$$

$$\nabla A = \frac{\partial A}{\partial x}\mathbf{i} + \frac{\partial A}{\partial y}\mathbf{j} + \frac{\partial A}{\partial z}\mathbf{k} = (2xyz + z^2)\mathbf{i} + x^2z\mathbf{j} + (x^2y + 2xz)\mathbf{k}$$

$$= 21\mathbf{i} + 12\mathbf{j} + 16\mathbf{k} \quad \text{at } (2, 1, 3)$$

Now $\nabla(AB) = A(\nabla B) + B(\nabla A) = \ldots\ldots\ldots\ldots$

Finish it

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$$\boxed{\nabla(AB) = 3(-117\mathbf{i} + 36\mathbf{j} - 362\mathbf{k})}$$

Because

$$\nabla(AB) = A(\nabla B) + B(\nabla A)$$

$$A = x^2yz + xz^2 \qquad \therefore \text{ at } (2, 1, 3), \qquad A = 12 + 18 = 30$$

$$B = xy^2z - z^3 \qquad \therefore \text{ at } (2, 1, 3), \qquad B = 6 - 27 = -21$$

$$\therefore \; \nabla(AB) = 30(3\mathbf{i} + 12\mathbf{j} - 25\mathbf{k}) - 21(21\mathbf{i} + 12\mathbf{j} + 16\mathbf{k})$$

$$= -351\mathbf{i} + 108\mathbf{j} - 1086\mathbf{k}$$

$$= 3(-117\mathbf{i} + 36\mathbf{j} - 362\mathbf{k})$$

So add these to the list of results.

$$\nabla(A + B) = \nabla A + \nabla B$$

$$\nabla(AB) = A(\nabla B) + B(\nabla A)$$

where A and B are scalars.

Now on to the next page

Div (divergence of a vector function)

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The operator $\nabla\cdot$ (notice the 'dot'; it makes all the difference) can be applied to a vector function $\mathbf{A}(x, y, z)$ to give the *divergence* of $\mathbf{A}$, written in short as *div* $\mathbf{A}$.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)\cdot\left(a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}\right)$$

$$\therefore\ \operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

Note that

(a) the grad operator ∇ acts on a scalar and gives a vector

(b) the div operation $\nabla\cdot$ acts on a vector and gives a scalar.

Example 1

If $\mathbf{A} = x^2y\mathbf{i} - xyz\mathbf{j} + yz^2\mathbf{k}$ then

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \dots\dots\dots\dots$$

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$$\boxed{\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = 2xy - xz + 2yz}$$

We simply take the appropriate partial derivatives of the coefficients of $\mathbf{i}$, $\mathbf{j}$ and $\mathbf{k}$. It could hardly be easier.

Example 2

If $\mathbf{A} = 2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$, determine $\nabla\cdot\mathbf{A}$, i.e. div $\mathbf{A}$.

Complete it. $\quad\nabla\cdot\mathbf{A} = \dots\dots\dots\dots$

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$$\boxed{\nabla\cdot\mathbf{A} = 0}$$

Because

$$\mathbf{A} = 2x^2y\mathbf{i} - 2(xy^2 + y^3z)\mathbf{j} + 3y^2z^2\mathbf{k}$$

$$\begin{aligned}\nabla\cdot\mathbf{A} &= \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}\\ &= 4xy - 2(2xy + 3y^2z) + 6y^2z\\ &= 4xy - 4xy - 6y^2z + 6y^2z = 0\end{aligned}$$

Such a vector $\mathbf{A}$ for which $\nabla\cdot\mathbf{A} = 0$ at all points, i.e. for all values of x, y, z, is called a *solenoidal vector*. It is rather a special case.

▶

Curl (curl of a vector function)

The *curl operator* denoted by $\nabla\times$, acts on a vector and gives another vector as a result.

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$, then curl $\mathbf{A} = \nabla \times \mathbf{A}$.

$$\text{i.e.}\quad \text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \times (a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k})$$

$$= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$\therefore\ \nabla \times \mathbf{A} = \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) + \mathbf{j}\left(\frac{\partial a_x}{\partial z} - \frac{\partial a_z}{\partial x}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

Curl $\mathbf{A}$ is thus a vector function. *It is best remembered in its determinant form, so make a note of it.*

If $\nabla \times \mathbf{A} = \mathbf{0}$ then $\mathbf{A}$ is said to be *irrotational*.

Then on for an example

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Example 1

If $\mathbf{A} = (y^4 - x^2z^2)\mathbf{i} + (x^2 + y^2)\mathbf{j} - x^2yz\mathbf{k}$, determine curl $\mathbf{A}$ at the point $(1,\ 3,\ -2)$.

$$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^4 - x^2z^2 & x^2 + y^2 & -x^2yz \end{vmatrix}$$

Now we expand the determinant

$$\nabla \times \mathbf{A} = \mathbf{i}\left\{\frac{\partial}{\partial y}(-x^2yz) - \frac{\partial}{\partial z}(x^2 + y^2)\right\} - \mathbf{j}\left\{\frac{\partial}{\partial x}(-x^2yz) - \frac{\partial}{\partial z}(y^4 - x^2z^2)\right\}$$
$$+ \mathbf{k}\left\{\frac{\partial}{\partial x}(x^2 + y^2) - \frac{\partial}{\partial y}(y^4 - x^2z^2)\right\}$$

All that now remains is to obtain the partial derivatives and substitute the values of x, y, z.

$$\therefore\ \nabla \times \mathbf{A} = \dots\dots\dots\dots$$

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$$\boxed{2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}}$$

$$\nabla \times \mathbf{A} = \mathbf{i}\{-x^2z\} - \mathbf{j}\{-2xyz + 2x^2z\} + \mathbf{k}\{2x - 4y^3\}.$$

$$\therefore\ \text{At } (1,\ 3,\ -2),\quad \nabla \times \mathbf{A} = \mathbf{i}(2) - \mathbf{j}(12 - 4) + \mathbf{k}(2 - 108)$$
$$= 2\mathbf{i} - 8\mathbf{j} - 106\mathbf{k}$$

Example 2

Determine curl **F** at the point (2, 0, 3) given that

$\mathbf{F} = ze^{2xy}\mathbf{i} + 2xz\cos y\mathbf{j} + (x + 2y)\mathbf{k}$.

In determinant form, curl $\mathbf{F} = \nabla \times \mathbf{F} = \ldots\ldots\ldots\ldots$

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$$\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ ze^{2xy} & 2xz\cos y & x + 2y \end{vmatrix}$$

Now expand the determinant and substitute the values of x, y and z, finally obtaining curl $\mathbf{F} = \ldots\ldots\ldots\ldots$

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$$\boxed{\text{curl } \mathbf{F} = \nabla \times \mathbf{F} = -2(\mathbf{i} + 3\mathbf{k})}$$

Because

$$\nabla \times \mathbf{F} = \mathbf{i}\{2 - 2x\cos y\} - \mathbf{j}\{1 - e^{2xy}\} + \mathbf{k}\{2z\cos y - 2xze^{2xy}\}$$

$$\therefore \text{ At } (2, 0, 3) \qquad \nabla \times \mathbf{F} = \mathbf{i}(2 - 4) - \mathbf{j}(1 - 1) + \mathbf{k}(6 - 12)$$

$$= -2\mathbf{i} - 6\mathbf{k} = -2(\mathbf{i} + 3\mathbf{k})$$

Every one is done in the same way.

Summary of grad, div and curl

(a) *Grad* operator ∇ acts on a *scalar* field to give a *vector* field.

(b) *Div* operator $\nabla\cdot$ acts on a *vector* field to give a *scalar* field.

(c) *Curl* operator $\nabla\times$ acts on a *vector* field to give a *vector* field.

(d) With a *scalar function* $\phi(x, y, z)$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

(e) With a *vector function* $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

(1) $\text{div}\,\mathbf{A} = \nabla\cdot\mathbf{A} = \dfrac{\partial a_x}{\partial x} + \dfrac{\partial a_y}{\partial y} + \dfrac{\partial a_z}{\partial z}$

(2) $\text{curl}\,\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$

Check through that list, just to make sure. We shall need them all

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By way of revision, here is one further example.

Example 3

If $\phi = x^2y^2 + x^3yz - yz^2$
and $\mathbf{F} = xy^2\mathbf{i} - 2yz\mathbf{j} + xyz\mathbf{k}$

determine for the point P (1, −1, 2),

(a) $\nabla\phi$, (b) unit normal, (c) $\nabla \cdot \mathbf{F}$, (d) $\nabla \times \mathbf{F}$.

Complete all four parts and then check the results with the next frame

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Here is the working in full. $\phi = x^2y^2 + x^3yz - yz^2$

(a) $\nabla\phi = \dfrac{\partial\phi}{\partial x}\mathbf{i} + \dfrac{\partial\phi}{\partial y}\mathbf{j} + \dfrac{\partial\phi}{\partial z}\mathbf{k}$

$= (2xy^2 + 3x^2yz)\mathbf{i} + (2x^2y + x^3z - z^2)\mathbf{j} + (x^3y - 2yz)\mathbf{k}$

$\therefore$ At (1, −1, 2) $\quad \nabla\phi = -4\mathbf{i} - 4\mathbf{j} + 3\mathbf{k}$

(b) $\mathbf{N} = \dfrac{\nabla\phi}{|\nabla\phi|} \qquad |\nabla\phi| = \sqrt{16+16+9} = \sqrt{41}$

$\therefore \mathbf{N} = \dfrac{-1}{\sqrt{41}}(4\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$

(c) $\mathbf{F} = xy^2\mathbf{i} - 2yz\mathbf{j} + xyz\mathbf{k} \qquad \nabla \cdot \mathbf{F} = \dfrac{\partial a_x}{\partial x} + \dfrac{\partial a_y}{\partial y} + \dfrac{\partial a_z}{\partial z}$

$\therefore \nabla \cdot \mathbf{F} = y^2 - 2z + xy$

$\therefore$ At (1, −1, 2) $\quad \nabla \cdot \mathbf{F} = 1 - 4 - 1 = -4 \qquad \therefore \nabla \cdot \mathbf{F} = -4$

(d) $$\nabla \times \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ xy^2 & -2yz & xyz \end{vmatrix}$$

$\therefore \nabla \times \mathbf{F} = \mathbf{i}(xz + 2y) - \mathbf{j}(yz - 0) + \mathbf{k}(0 - 2xy)$

$= (xz + 2y)\mathbf{i} - yz\mathbf{j} - 2xy\mathbf{k}$

$\therefore$ At (1, −1, 2) $\quad \nabla \times \mathbf{F} = 2\mathbf{j} + 2\mathbf{k} \qquad \therefore \nabla \times \mathbf{F} = 2(\mathbf{j} + \mathbf{k})$

Now let us combine some of these operations.

Multiple operations

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We can combine the operators grad, div and curl in multiple operations, as in the examples that follow.

Example 1

If $\mathbf{A} = x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k}$

$$\text{then} \quad \text{div}\,\mathbf{A} = \nabla \cdot \mathbf{A} = \left(\frac{\partial}{\partial x}\mathbf{i} + \frac{\partial}{\partial y}\mathbf{j} + \frac{\partial}{\partial z}\mathbf{k}\right) \cdot (x^2y\mathbf{i} + yz^3\mathbf{j} - zx^3\mathbf{k})$$

$$= 2xy + z^3 + x^3 = \phi \quad \text{say}$$

$$\text{Then} \quad \text{grad}\,(\text{div}\,\mathbf{A}) = \nabla(\nabla \cdot \mathbf{A}) = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$= (2y + 3x^2)\mathbf{i} + (2x)\mathbf{j} + (3z^2)\mathbf{k}$$

$$\text{i.e.} \quad \text{grad div}\,\mathbf{A} = \nabla(\nabla \cdot \mathbf{A}) = (2y + 3x^2)\mathbf{i} + 2x\mathbf{j} + 3z^2\mathbf{k}$$

Move on for the next example

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Example 2

If $\phi = xyz - 2y^2z + x^2z^2$, determine div grad ϕ at the point (2, 4, 1).

First find grad ϕ and then the div of the result.

At (2, 4, 1), div grad $\phi = \nabla \cdot (\nabla\phi) = \dots\dots\dots\dots$

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$$\boxed{\text{div grad}\,\phi = 6}$$

Because we have $\phi = xyz - 2y^2z + x^2z^2$

$$\text{grad}\,\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$= (yz + 2xz^2)\mathbf{i} + (xz - 4yz)\mathbf{j} + (xy - 2y^2 + 2x^2z)\mathbf{k}$$

$$\therefore \ \text{div grad}\,\phi = \nabla \cdot (\nabla\phi) = 2z^2 - 4z + 2x^2$$

$$\therefore \ \text{At (2, 4, 1)}, \ \text{div grad}\,\phi = \nabla \cdot (\nabla\phi) = 2 - 4 + 8 = 6$$

Example 3

If $\mathbf{F} = x^2yz\mathbf{i} + xyz^2\mathbf{j} + y^2z\mathbf{k}$ determine curl curl $\mathbf{F}$ at the point (2, 1, 1).

Determine an expression for curl $\mathbf{F}$ in the usual way, which will be a vector, and then the curl of the result. Finally substitute values.

curl curl $\mathbf{F} = \dots\dots\dots\dots$

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$$\boxed{\text{curl curl } \mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}}$$

Because

$$\text{curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x^2yz & xyz^2 & y^2z \end{vmatrix}$$

$$= (2yz - 2xyz)\mathbf{i} + x^2y\mathbf{j} + (yz^2 - x^2z)\mathbf{k}$$

Then $\quad \text{curl curl } \mathbf{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ 2yz - 2xyz & x^2y & yz^2 - x^2z \end{vmatrix}$

$$= z^2\mathbf{i} - (-2xz - 2y + 2xy)\mathbf{j} + (2xy - 2z + 2xz)\,\mathbf{k}$$

$\therefore$ At (2, 1, 1), $\quad \text{curl curl } \mathbf{F} = \nabla \times (\nabla \times \mathbf{F}) = \mathbf{i} + 2\mathbf{j} + 6\mathbf{k}$

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Remember that grad, div and curl are operators and that they must act on a scalar or vector as appropriate. They cannot exist alone and must be followed by a function.

One or two interesting general results appear.

(a) *Curl grad* ϕ where ϕ is a scalar

$$\text{grad } \phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$\therefore \text{ curl grad } \phi = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \dfrac{\partial \phi}{\partial x} & \dfrac{\partial \phi}{\partial y} & \dfrac{\partial \phi}{\partial z} \end{vmatrix}$$

$$= \mathbf{i}\left\{\frac{\partial^2 \phi}{\partial y \partial z} - \frac{\partial^2 \phi}{\partial z \partial y}\right\} - \mathbf{j}\left\{\frac{\partial^2 \phi}{\partial z \partial x} - \frac{\partial^2 \phi}{\partial x \partial z}\right\} + \mathbf{k}\left\{\frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x}\right\}$$

$$= 0$$

$$\therefore \text{ curl grad } \phi = \nabla \times (\nabla \phi) = 0$$

▶

(b) *Div curl* **A** where **A** is a vector. $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$

$$\text{curl}\,\mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

$$= \mathbf{i}\left(\frac{\partial a_z}{\partial y} - \frac{\partial a_y}{\partial z}\right) - \mathbf{j}\left(\frac{\partial a_z}{\partial x} - \frac{\partial a_x}{\partial z}\right) + \mathbf{k}\left(\frac{\partial a_y}{\partial x} - \frac{\partial a_x}{\partial y}\right)$$

$$\text{Then div curl}\,\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot (\nabla \times \mathbf{A})$$

$$= \frac{\partial^2 a_z}{\partial x \partial y} - \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_z}{\partial x \partial y} + \frac{\partial^2 a_x}{\partial y \partial z} + \frac{\partial^2 a_y}{\partial z \partial x} - \frac{\partial^2 a_x}{\partial y \partial z}$$

$$= 0$$

$$\therefore\ \text{div curl}\,\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = \mathbf{0}$$

(c) *Div grad* ϕ where ϕ is a scalar

$$\text{grad}\,\phi = \frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}$$

$$\text{Then div grad}\,\phi = \nabla \cdot (\nabla \phi)$$

$$= \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right) \cdot \left(\frac{\partial \phi}{\partial x}\mathbf{i} + \frac{\partial \phi}{\partial y}\mathbf{j} + \frac{\partial \phi}{\partial z}\mathbf{k}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$\therefore\ \text{div grad}\,\phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \nabla^2 \phi, \text{ the Laplacian of } \phi$$

The operator ∇^2 is called the Laplacian.

So these general results are

(a) curl grad $\phi = \nabla \times (\nabla \phi) = 0$

(b) div curl $\mathbf{A} = \nabla \cdot (\nabla \times \mathbf{A}) = 0$

(c) div grad $\phi = \nabla \cdot (\nabla \phi) = \dfrac{\partial^2 \phi}{\partial x^2} + \dfrac{\partial^2 \phi}{\partial y^2} + \dfrac{\partial^2 \phi}{\partial z^2}$.

That brings us to the end of this particular Programme. We have covered quite a lot of new material, so check carefully through the **Revision summary** and **Can You?** checklist that follow: then you can deal with the **Test exercise**. The **Further problems** provide an opportunity for additional practice.

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Revision summary 17

If $\mathbf{A} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k}$; $\mathbf{B} = b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}$; $\mathbf{C} = c_x\mathbf{i} + c_y\mathbf{j} + c_z\mathbf{k}$; then we have the following relationships.

1 *Scalar product* (dot product) $\quad \mathbf{A}\cdot\mathbf{B} = AB\cos\theta$

$$\mathbf{A}\cdot\mathbf{B} = \mathbf{B}\cdot\mathbf{A} \quad \text{and} \quad \mathbf{A}\cdot(\mathbf{B}+\mathbf{C}) = \mathbf{A}\cdot\mathbf{B} + \mathbf{A}\cdot\mathbf{C}$$

If $\mathbf{A}\cdot\mathbf{B} = 0$ and $\mathbf{A}$, $\mathbf{B} \neq \mathbf{0}$ then $\mathbf{A} \perp \mathbf{B}$.

2 *Vector product* (cross product) $\quad \mathbf{A}\times\mathbf{B} = (AB\sin\theta)\mathbf{n}$

$\mathbf{n}$ = unit normal vector where $\mathbf{A}$, $\mathbf{B}$, $\mathbf{n}$ form a right-handed set.

$$\mathbf{A}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$$

$\mathbf{A}\times\mathbf{B} = -(\mathbf{B}\times\mathbf{A})$ and $\mathbf{A}\times(\mathbf{B}+\mathbf{C}) = \mathbf{A}\times\mathbf{B} + \mathbf{A}\times\mathbf{C}$

3 *Unit vectors*

(a) $\mathbf{i}\cdot\mathbf{i} = \mathbf{j}\cdot\mathbf{j} = \mathbf{k}\cdot\mathbf{k} = 1$
$\mathbf{i}\cdot\mathbf{j} = \mathbf{j}\cdot\mathbf{k} = \mathbf{k}\cdot\mathbf{i} = 0.$

(b) $\mathbf{i}\times\mathbf{i} = \mathbf{j}\times\mathbf{j} = \mathbf{k}\times\mathbf{k} = \mathbf{0}$
$\mathbf{i}\times\mathbf{j} = \mathbf{k}, \quad \mathbf{j}\times\mathbf{k} = \mathbf{i}, \quad \mathbf{k}\times\mathbf{i} = \mathbf{j}.$

4 *Scalar triple product* $\quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C})$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \begin{vmatrix} a_x & a_y & a_z \\ b_x & b_y & b_z \\ c_x & c_y & c_z \end{vmatrix}$$

$$\mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = \mathbf{B}\cdot(\mathbf{C}\times\mathbf{A}) = \mathbf{C}\cdot(\mathbf{A}\times\mathbf{B})$$

Unchanged by cyclic change of vectors.
Sign reversed by non-cyclic change of vectors.

5 *Coplanar vectors* $\quad \mathbf{A}\cdot(\mathbf{B}\times\mathbf{C}) = 0.$

6 *Vector triple product* $\quad \mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ and $(\mathbf{A}\times\mathbf{B})\times\mathbf{C}$

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

and $\quad (\mathbf{A}\times\mathbf{B})\times\mathbf{C} = (\mathbf{C}\cdot\mathbf{A})\mathbf{B} - (\mathbf{C}\cdot\mathbf{B})\mathbf{A}.$

7 *Differentiation of vectors*

If $\mathbf{A}$, a_x, a_y, a_z are functions of u

$$\frac{d\mathbf{A}}{du} = \frac{da_x}{du}\mathbf{i} + \frac{da_y}{du}\mathbf{j} + \frac{da_z}{du}\mathbf{k}$$

8 *Unit tangent vector* $\mathbf{T}$

$$\mathbf{T} = \frac{\dfrac{d\mathbf{A}}{du}}{\left|\dfrac{d\mathbf{A}}{du}\right|}$$

▶

9 *Integration of vectors*

$$\int_a^b \mathbf{A}\,\mathrm{d}u = \mathbf{i}\int_a^b a_x\,\mathrm{d}u + \mathbf{j}\int_a^b a_y\,\mathrm{d}u + \mathbf{k}\int_a^b a_z\,\mathrm{d}u$$

10 *Grad* (gradient of a scalar function ϕ)

$$\operatorname{grad}\phi = \nabla\phi = \frac{\partial\phi}{\partial x}\mathbf{i} + \frac{\partial\phi}{\partial y}\mathbf{j} + \frac{\partial\phi}{\partial z}\mathbf{k}$$

$$\text{'del'} = \text{operator } \nabla = \left(\mathbf{i}\frac{\partial}{\partial x} + \mathbf{j}\frac{\partial}{\partial y} + \mathbf{k}\frac{\partial}{\partial z}\right)$$

(a) *Directional derivative* $\dfrac{\mathrm{d}\phi}{\mathrm{d}s} = \hat{\mathbf{a}}\cdot\operatorname{grad}\phi = \hat{\mathbf{a}}\cdot\nabla\phi$ where $\hat{\mathbf{a}}$ is a unit vector in a stated direction. Grad ϕ gives the direction for maximum rate of change of ϕ.

(b) *Unit normal vector* **N** to surface $\phi(x, y, z) = \text{constant}$.

$$\mathbf{N} = \frac{\nabla\phi}{|\nabla\phi|}$$

11 *Div* (divergence of a vector function **A**)

$$\operatorname{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z}$$

If $\nabla\cdot\mathbf{A} = 0$ for all points, **A** is a solenoidal vector.

12 *Curl* (curl of a vector function **A**)

$$\operatorname{curl}\mathbf{A} = \nabla\times\mathbf{A} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ a_x & a_y & a_z \end{vmatrix}$$

If $\nabla\times\mathbf{A} = 0$ then **A** is an irrotational vector.

13 *Operators*

grad (∇) acts on a *scalar* and gives a *vector*

div ($\nabla\cdot$) acts on a *vector* and gives a *scalar*

curl ($\nabla\times$) acts on a *vector* and gives a *vector.*

14 *Multiple operations*

(a) curl grad $\phi = \nabla\times(\nabla\phi) = 0$

(b) div curl $\mathbf{A} = \nabla\cdot(\nabla\times\mathbf{A}) = 0$

(c) div grad $\phi = \nabla\cdot(\nabla\phi) = \dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} + \dfrac{\partial^2\phi}{\partial z^2}$

$= \nabla^2\phi$, the Laplacian of ϕ.

Can You?

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Checklist 17

Check this list before and after you try the end of Programme test.

On a scale of 1 to 5 how confident are you that you can: Frames

- Obtain the scalar and vector product of two vectors? 1 to 4

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Reproduce the relationships between the scalar and vector products of the Cartesian coordinate unit vectors? 5 to 11

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the scalar and vector triple products and appreciate their geometric significance? 12 to 27

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Differentiate a vector field and derive a unit vector tangential to the vector field at a point? 28 to 48

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Integrate a vector field? 49 to 55

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the gradient of a scalar field, the directional derivative and a unit normal to a surface? 56 to 77

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the divergence of a vector field and recognise a solenoidal vector field? 78 to 80

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain the curl of a vector field? 80 to 86

  Yes ☐ ☐ ☐ ☐ ☐ *No*

- Obtain combinations of div, grad and curl acting on scalar and vector fields as appropriate? 87 to 91

  Yes ☐ ☐ ☐ ☐ ☐ *No*

Test exercise 17

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1 Find (a) the scalar product and (b) the vector product of the vectors $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + 4\mathbf{k}$ and $\mathbf{B} = \mathbf{i} + 5\mathbf{j} - 2\mathbf{k}$.

2 If $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 5\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = \mathbf{i} - \mathbf{j} + 3\mathbf{k}$; determine

(a) the scalar triple product $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$

(b) the vector triple product $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

3 Determine whether the three vectors $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} + \mathbf{k}$; $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + \mathbf{j} + 3\mathbf{k}$ are coplanar.

4 If $\mathbf{A} = (u^2 + 5)\mathbf{i} - (u^2 + 3)\mathbf{j} + 2u^3\mathbf{k}$, determine

(a) $\dfrac{d\mathbf{A}}{du}$; (b) $\dfrac{d^2\mathbf{A}}{du^2}$; (c) $\left|\dfrac{d\mathbf{A}}{du}\right|$; all at $u = 2$.

5 Determine the unit tangent vector at the point (2, 4, 3) for the curve with parametric equations

$x = 2u^2$; $y = u + 3$; $z = 4u^2 - u$.

6 If $\mathbf{F} = 2\mathbf{i} + 4u\mathbf{j} + u^2\mathbf{k}$ and $\mathbf{G} = u^2\mathbf{i} - 2u\mathbf{j} + 4\mathbf{k}$, determine

$\displaystyle\int_0^2 (\mathbf{F} \times \mathbf{G})du$.

7 Find the directional derivative of the function $\phi = x^2y - 2xz^2 + y^2z$ at the point (1, 3, 2) in the direction of the vector $\mathbf{A} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$.

8 Find the unit normal to the surface $\phi = 2x^3z + x^2y^2 + xyz - 4 = 0$ at the point (2, 1, 0).

9 If $\mathbf{A} = x^2y\mathbf{i} + (xy + yz)\mathbf{j} + xz^2\mathbf{k}$; $\mathbf{B} = yz\mathbf{i} - 3xz\mathbf{j} + 2xy\mathbf{k}$; and $\phi = 3x^2y + xyz - 4y^2z^2 - 3$; determine, at the point (1, 2, 1)

(a) $\nabla\phi$; (b) $\nabla \cdot \mathbf{A}$; (c) $\nabla \times \mathbf{B}$; (d) grad div $\mathbf{A}$; (e) curl curl $\mathbf{A}$.

Further problems 17

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1 If $\mathbf{A} = 2\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + 5\mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$; determine $\mathbf{A} \cdot (\mathbf{B} \times \mathbf{C})$.

2 If $\mathbf{A} = 2\mathbf{i} + \mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = \mathbf{i} - 2\mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + 2\mathbf{j} - \mathbf{k}$; find $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$.

3 If $\mathbf{A} = \mathbf{i} - 2\mathbf{j} + 3\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + 2\mathbf{j} + \mathbf{k}$; find

(a) $\mathbf{A} \times (\mathbf{B} \times \mathbf{C})$; (b) $(\mathbf{A} \times \mathbf{B}) \times \mathbf{C}$.

4 If $\mathbf{F} = x^2\mathbf{i} + (3x + 2)\mathbf{j} + \sin x\mathbf{k}$, find

(a) $\dfrac{d\mathbf{F}}{dx}$; (b) $\dfrac{d^2\mathbf{F}}{dx^2}$; (c) $\left|\dfrac{d\mathbf{F}}{dx}\right|$; (d) $\dfrac{d}{dx}(\mathbf{F} \cdot \mathbf{F})$ at $x = 1$.

▶

5 If $\mathbf{F} = u\mathbf{i} + (1-u)\mathbf{j} + 3u\mathbf{k}$ and $\mathbf{G} = 2\mathbf{i} - (1+u)\mathbf{j} - u^2\mathbf{k}$, determine
(a) $\dfrac{d}{du}(\mathbf{F}\cdot\mathbf{G})$; (b) $\dfrac{d}{du}(\mathbf{F}\times\mathbf{G})$; (c) $\dfrac{d}{du}(\mathbf{F}+\mathbf{G})$.

6 Find the unit normal to the surface $4x^2y^2 - 3xz^2 - 2y^2z + 4 = 0$ at the point $(2, -1, -2)$.

7 Find the unit normal to the surface $2xy^2 + y^2z + x^2z - 11 = 0$ at the point $(-2, 1, 3)$.

8 Determine the unit vector normal to the surface
$xz^2 + 3xy - 2yz^2 + 1 = 0$ at the point $(1, -2, -1)$.

9 Find the unit normal to the surface $x^2y - 2yz^2 + y^2z = 3$ at the point $(2, -3, 1)$.

10 Determine the directional derivative of $\phi = xe^y + yz^2 + xyz$ at the point $(2, 0, 3)$ in the direction of $\mathbf{A} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.

11 Find the directional derivative of $\phi = (x+2y+z)^2 - (x-y-z)^2$ at the point $(2, 1, -1)$ in the direction of $\mathbf{A} = \mathbf{i} - 4\mathbf{j} + 2\mathbf{k}$.

12 Find the scalar triple product of
(a) $\mathbf{A} = \mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} - \mathbf{j} + 4\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$.
(b) $\mathbf{A} = 2\mathbf{i} - 3\mathbf{j} + \mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + \mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = \mathbf{i} + 4\mathbf{j} - 2\mathbf{k}$.
(c) $\mathbf{A} = -2\mathbf{i} + 3\mathbf{j} - 2\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} - \mathbf{j} + 3\mathbf{k}$; $\mathbf{C} = 2\mathbf{i} - 5\mathbf{j} + \mathbf{k}$.

13 Find the vector triple product $\mathbf{A}\times(\mathbf{B}\times\mathbf{C})$ of the following.
(a) $\mathbf{A} = 3\mathbf{i} + \mathbf{j} - 2\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} + 4\mathbf{j} + 3\mathbf{k}$; $\mathbf{C} = \mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
(b) $\mathbf{A} = 2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$; $\mathbf{B} = \mathbf{i} + 4\mathbf{j} - 5\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} - 2\mathbf{j} + \mathbf{k}$.
(c) $\mathbf{A} = 4\mathbf{i} + 2\mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = 2\mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$; $\mathbf{C} = 3\mathbf{i} - 3\mathbf{j} + \mathbf{k}$.

14 If $\mathbf{F} = 4t^3\mathbf{i} - 2t^2\mathbf{j} + 4t\mathbf{k}$, determine when $t = 1$
(a) $\dfrac{d\mathbf{F}}{dt}$; (b) $\dfrac{d^2\mathbf{F}}{dt^2}$; (c) $\dfrac{d}{dt}(\mathbf{F}\cdot\mathbf{F})$.

15 If $\phi = x^2\sin z + ze^y$ find, at the point $(1, 3, 2)$, the values of
(a) grad ϕ and (b) $|\text{grad } \phi|$.

16 Given that $\phi = xy^2 + yz^2 - x^2$, find the derivative of ϕ with respect to distance at the point $(1, 2, -1)$, measured parallel to the vector $2\mathbf{i} - 3\mathbf{j} + 4\mathbf{k}$.

17 Find unit vectors normal to the surfaces $x^2 + y^2 - z^2 + 3 = 0$ and $xy - yz + zx - 10 = 0$ at the point $(3, 2, 4)$ and hence find the angle between the two surfaces at that point.

18 If $\mathbf{r} = (t^2 + 3t)\mathbf{i} - 2\sin 3t\mathbf{j} + 3e^{2t}\mathbf{k}$, determine
(a) $\dfrac{d\mathbf{r}}{dt}$; (b) $\dfrac{d^2\mathbf{r}}{dt^2}$; (c) the value of $\left|\dfrac{d^2\mathbf{r}}{dt^2}\right|$ at $t = 0$.

▶

19 (a) Show that curl $(-y\mathbf{i} + x\mathbf{j})$ is a constant vector.

(b) Show that the vector field $(yz\mathbf{i} + zx\mathbf{j} + xy\mathbf{k})$ has zero divergence and zero curl.

20 If $\mathbf{A} = 2xz^2\mathbf{i} - xz\mathbf{j} + (y + z)\mathbf{k}$, find curl curl $\mathbf{A}$.

21 Determine grad ϕ where $\phi = x^2 \cos(2yz - 0{\cdot}5)$ and obtain its value at the point (1, 3, 1).

22 Determine the value of p such that the three vectors **A**, **B**, **C** are coplanar when $\mathbf{A} = 2\mathbf{i} + \mathbf{j} + 4\mathbf{k}$; $\mathbf{B} = 3\mathbf{i} + 2\mathbf{j} + p\mathbf{k}$; $\mathbf{C} = \mathbf{i} + 4\mathbf{j} + 2\mathbf{k}$.

23 If $\mathbf{A} = p\mathbf{i} - 6\mathbf{j} - 3\mathbf{k}$; $\mathbf{B} = 4\mathbf{i} + 3\mathbf{j} - \mathbf{k}$; $\mathbf{C} = \mathbf{i} - 3\mathbf{j} + 2\mathbf{k}$

(a) find the values of p for which

(1) **A** and **B** are perpendicular to each other

(2) **A**, **B** and **C** are coplanar.

(b) determine a unit vector perpendicular to both **A** and **B** when $p = 2$.